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GTM

Sheldon Axler

Measure, Integration & Real Analysis

$$\int |fg|\,d\mu \le \left(\int |f|^p\,d\mu\right)^{1/p}\left(\int |g|^{p'}\,d\mu\right)^{1/p'}$$

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Dedicated to

Paul Halmos, Don Sarason, and Allen Shields,

the three mathematicians who most
helped me become a mathematician.

About the Author

Sheldon Axler was valedictorian of his high school in Miami, Florida. He received his AB from Princeton University with highest honors, followed by a PhD in Mathematics from the University of California at Berkeley.

As a postdoctoral Moore Instructor at MIT, Axler received a university-wide teaching award. He was then an assistant professor, associate professor, and professor at Michigan State University, where he received the first J. Sutherland Frame Teaching Award and the Distinguished Faculty Award.

Axler received the Lester R. Ford Award for expository writing from the Mathematical Association of America in 1996. In addition to publishing numerous research papers, he is the author of six mathematics textbooks, ranging from freshman to graduate level. His book *Linear Algebra Done Right* has been adopted as a textbook at over 300 universities and colleges.

Axler has served as Editor-in-Chief of the *Mathematical Intelligencer* and Associate Editor of the *American Mathematical Monthly*. He has been a member of the Council of the American Mathematical Society and a member of the Board of Trustees of the Mathematical Sciences Research Institute. He has also served on the editorial board of Springer's series Undergraduate Texts in Mathematics, Graduate Texts in Mathematics, Universitext, and Springer Monographs in Mathematics.

He has been honored by appointments as a Fellow of the American Mathematical Society and as a Senior Fellow of the California Council on Science and Technology.

Axler joined San Francisco State University as Chair of the Mathematics Department in 1997. In 2002, he became Dean of the College of Science & Engineering at San Francisco State University. After serving as Dean for thirteen years, he returned to a regular faculty appointment as a professor in the Mathematics Department.



Cover figure: Hölder's Inequality, which is proved in Section 7A.

Contents

Preface for Students

You are about to immerse yourself in serious mathematics, with an emphasis on attaining a deep understanding of the definitions, theorems, and proofs related to measure, integration, and real analysis. This book aims to guide you to the wonders of this subject.

You cannot read mathematics the way you read a novel. If you zip through a page in less than an hour, you are probably going too fast. When you encounter the phrase *as you should verify*, you should indeed do the verification, which will usually require some writing on your part. When steps are left out, you need to supply the missing pieces. You should ponder and internalize each definition. For each theorem, you should seek examples to show why each hypothesis is necessary.

Working on the exercises should be your main mode of learning after you have read a section. Discussions and joint work with other students may be especially effective. Active learning promotes long-term understanding much better than passive learning. Thus you will benefit considerably from struggling with an exercise and eventually coming up with a solution, perhaps working with other students. Finding and reading a solution on the internet will likely lead to little learning.

As a visual aid, throughout this book definitions are in yellow boxes and theorems are in blue boxes, in both print and electronic versions. Each theorem has an informal descriptive name. The electronic version of this manuscript has links in blue.

Please check the website below (or the Springer website) for additional information about the book. These websites link to the electronic version of this book, which is free to the world because this book has been published under Springer's Open Access program. Your suggestions for improvements and corrections for a future edition are most welcome (send to the email address below).

The prerequisite for using this book includes a good understanding of elementary undergraduate real analysis. You can download from the website below or from the Springer website the document titled *Supplement for Measure, Integration & Real Analysis*. That supplement can serve as a review of the elementary undergraduate real analysis used in this book.

Best wishes for success and enjoyment in learning measure, integration, and real analysis!

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Preface for Instructors

You are about to teach a course, or possibly a two-semester sequence of courses, on measure, integration, and real analysis. In this textbook, I have tried to use a gentle approach to serious mathematics, with an emphasis on students attaining a deep understanding. Thus new material often appears in a comfortable context instead of the most general setting. For example, the Fourier transform in Chapter 11 is introduced in the setting of $\mathbf{R}$ rather than $\mathbf{R}^n$ so that students can focus on the main ideas without the clutter of the extra bookkeeping needed for working in $\mathbf{R}^n$.

The basic prerequisite for your students to use this textbook is a good understanding of elementary undergraduate real analysis. Your students can download from the book's website (http://measure.axler.net) or from the Springer website the document titled *Supplement for Measure, Integration & Real Analysis*. That supplement can serve as a review of the elementary undergraduate real analysis used in this book.

As a visual aid, throughout this book definitions are in yellow boxes and theorems are in blue boxes, in both print and electronic versions. Each theorem has an informal descriptive name. The electronic version of this manuscript has links in blue.

Mathematics can be learned only by doing. Fortunately, real analysis has many good homework exercises. When teaching this course, during each class I usually assign as homework several of the exercises, due the next class. I grade only one exercise per homework set, but the students do not know ahead of time which one. I encourage my students to work together on the homework or to come to me for help. However, I tell them that getting solutions from the internet is not allowed and would be counterproductive for their learning goals.

If you go at a leisurely pace, then covering Chapters 1–5 in the first semester may be a good goal. If you go a bit faster, then covering Chapters 1–6 in the first semester may be more appropriate. For a second-semester course, covering some subset of Chapters 6 through 12 should produce a good course. Most instructors will not have time to cover all those chapters in a second semester; thus some choices need to be made. The following chapter-by-chapter summary of the highlights of the book should help you decide what to cover and in what order:

- **Chapter 1:** This short chapter begins with a brief review of Riemann integration. Then a discussion of the deficiencies of the Riemann integral helps motivate the need for a better theory of integration.

- **Chapter 2:** This chapter begins by defining outer measure on $\mathbf{R}$ as a natural extension of the length function on intervals. After verifying some nice properties of outer measure, we see that it is not additive. This observation leads to restricting our attention to the σ-algebra of Borel sets, defined as the smallest σ-algebra on $\mathbf{R}$ containing all the open sets. This path leads us to measures.

After dealing with the properties of general measures, we come back to the setting of **R**, showing that outer measure restricted to the σ-algebra of Borel sets is countably additive and thus is a measure. Then a subset of **R** is defined to be Lebesgue measurable if it differs from a Borel set by a set of outer measure 0. This definition makes Lebesgue measurable sets seem more natural to students than the other competing equivalent definitions. The Cantor set and the Cantor function then stretch students' intuition.

Egorov's Theorem, which states that pointwise convergence of a sequence of measurable functions is close to uniform convergence, has multiple applications in later chapters. Luzin's Theorem, back in the context of **R**, sounds spectacular but has no other uses in this book and thus can be skipped if you are pressed for time.

- **Chapter 3:** Integration with respect to a measure is defined in this chapter in a natural fashion first for nonnegative measurable functions, and then for real-valued measurable functions. The Monotone Convergence Theorem and the Dominated Convergence Theorem are the big results in this chapter that allow us to interchange integrals and limits under appropriate conditions.

- **Chapter 4:** The highlight of this chapter is the Lebesgue Differentiation Theorem, which allows us to differentiate an integral. The main tool used to prove this result cleanly is the Hardy–Littlewood maximal inequality, which is interesting and important in its own right. This chapter also includes the Lebesgue Density Theorem, showing that a Lebesgue measurable subset of **R** has density 1 at almost every number in the set and density 0 at almost every number not in the set.

- **Chapter 5:** This chapter deals with product measures. The most important results here are Tonelli's Theorem and Fubini's Theorem, which allow us to evaluate integrals with respect to product measures as iterated integrals and allow us to change the order of integration under appropriate conditions. As an application of product measures, we get Lebesgue measure on $\mathbf{R}^n$ from Lebesgue measure on **R**. To give students practice with using these concepts, this chapter finds a formula for the volume of the unit ball in $\mathbf{R}^n$. The chapter closes by using Fubini's Theorem to give a simple proof that a mixed partial derivative with sufficient continuity does not depend upon the order of differentiation.

- **Chapter 6:** After a quick review of metric spaces and vector spaces, this chapter defines normed vector spaces. The big result here is the Hahn–Banach Theorem about extending bounded linear functionals from a subspace to the whole space. Then this chapter introduces Banach spaces. We see that completeness plays a major role in the key theorems: Open Mapping Theorem, Inverse Mapping Theorem, Closed Graph Theorem, and Principle of Uniform Boundedness.

- **Chapter 7:** This chapter introduces the important class of Banach spaces $L^p(\mu)$, where $1 \leq p \leq \infty$ and μ is a measure, giving students additional opportunities to use results from earlier chapters about measure and integration theory. The crucial results called Hölder's inequality and Minkowski's inequality are key tools here. This chapter also shows that the dual of ℓ^p is $\ell^{p'}$ for $1 \leq p < \infty$.

  Chapters 1 through 7 should be covered in order, before any of the later chapters. After Chapter 7, you can cover Chapter 8 or Chapter 12.

- **Chapter 8:** This chapter focuses on Hilbert spaces, which play a central role in modern mathematics. After proving the Cauchy–Schwarz inequality and the Riesz Representation Theorem that describes the bounded linear functionals on a Hilbert space, this chapter deals with orthonormal bases. Key results here include Bessel's inequality, Parseval's identity, and the Gram–Schmidt process.

- **Chapter 9:** Only positive measures have been discussed in the book up until this chapter. In this chapter, real and complex measures get consideration. These concepts lead to the Banach space of measures, with total variation as the norm. Key results that help describe real and complex measures are the Hahn Decomposition Theorem, the Jordan Decomposition Theorem, and the Lebesgue Decomposition Theorem. The Radon–Nikodym Theorem is proved using von Neumann's slick Hilbert space trick. Then the Radon–Nikodym Theorem is used to prove that the dual of $L^p(\mu)$ can be identified with $L^{p'}(\mu)$ for $1 < p < \infty$ and μ a (positive) measure, completing a project that started in Chapter 7.

  The material in Chapter 9 is not used later in the book. Thus this chapter can be skipped or covered after one of the later chapters.

- **Chapter 10:** This chapter begins by discussing the adjoint of a bounded linear map between Hilbert spaces. Then the rest of the chapter presents key results about bounded linear operators from a Hilbert space to itself. The proof that each bounded operator on a complex nonzero Hilbert space has a nonempty spectrum requires a tiny bit of knowledge about analytic functions. Properties of special classes of operators (self-adjoint operators, normal operators, isometries, and unitary operators) are described.

  Then this chapter delves deeper into compact operators, proving the Fredholm Alternative. The chapter concludes with two major results: the Spectral Theorem for compact operators and the popular Singular Value Decomposition for compact operators. Throughout this chapter, the Volterra operator is used as an example to illustrate the main results.

  Some instructors may prefer to cover Chapter 10 immediately after Chapter 8, because both chapters live in the context of Hilbert space. I chose the current order to give students a breather between the two Hilbert space chapters, thinking that being away from Hilbert space for a little while and then coming back to it might strengthen students' understanding and provide some variety. However, covering the two Hilbert space chapters consecutively would also work fine.

- **Chapter 11:** Fourier analysis is a huge subject with a two-hundred year history. This chapter gives a gentle but modern introduction to Fourier series and the Fourier transform.

  This chapter first develops results in the context of Fourier series, but then comes back later and develops parallel concepts in the context of the Fourier transform. For example, the Fourier coefficient version of the Riemann–Lebesgue Lemma is proved early in the chapter, with the Fourier transform version proved later in the chapter. Other examples include the Poisson kernel, convolution, and the Dirichlet problem, all of which are first covered in the context of the unit disk and unit circle; then these topics are revisited later in the context of the half-plane and real line.

Convergence of Fourier series is proved in the L^2 norm and also (for sufficiently smooth functions) pointwise. The book emphasizes getting students to work with the main ideas rather than on proving all possible results (for example, pointwise convergence of Fourier series is proved only for twice continuously differentiable functions rather than using a weaker hypothesis).

The proof of the Fourier Inversion Formula is the highlight of the material on the Fourier transform. The Fourier Inversion Formula is then used to show that the Fourier transform extends to a unitary operator on $L^2(\mathbf{R})$.

This chapter uses some basic results about Hilbert spaces, so it should not be covered before Chapter 8. However, if you are willing to skip or hand-wave through one result that helps describe the Fourier transform as an operator on $L^2(\mathbf{R})$ (see 11.87), then you could cover this chapter without doing Chapter 10.

- **Chapter 12:** A thorough coverage of probability theory would require a whole book instead of a single chapter. This chapter takes advantage of the book's earlier development of measure theory to present the basic language and emphasis of probability theory. For students not pursuing further studies in probability theory, this chapter gives them a good taste of the subject. Students who go on to learn more probability theory should benefit from the head start provided by this chapter and the background of measure theory.

  Features that distinguish probability theory from measure theory include the notions of independent events and independent random variables. In addition to those concepts, this chapter discusses standard deviation, conditional probabilities, Bayes' Theorem, and distribution functions. The chapter concludes with a proof of the Weak Law of Large Numbers for independent identically distributed random variables.

  You could cover this chapter anytime after Chapter 7.

Please check the website below (or the Springer website) for additional information about the book. These websites link to the electronic version of this book, which is free to the world because this book has been published under Springer's Open Access program. Your suggestions for improvements and corrections for a future edition are most welcome (send to the email address below).

I enjoy keeping track of where my books are used as textbooks. If you use this book as the textbook for a course, please let me know.

Best wishes for teaching a successful class on measure, integration, and real analysis!

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Acknowledgments

I owe a huge intellectual debt to the many mathematicians who created real analysis over the past several centuries. The results in this book belong to the common heritage of mathematics. A special case of a theorem may first have been proved by one mathematician and then sharpened and improved by many other mathematicians. Bestowing accurate credit for all the contributions would be a difficult task that I have not undertaken. In no case should the reader assume that any theorem presented here represents my original contribution. However, in writing this book I tried to think about the best way to present real analysis and to prove its theorems, without regard to the standard methods and proofs used in most textbooks.

The manuscript for this book received an unusually large amount of class testing at several universities before publication. Thus I received many valuable suggestions for improvements and corrections. I am deeply grateful to all the faculty and students who helped class test the manuscript. I implemented suggestions or corrections from the following faculty and students, all of whom helped make this a better book:

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Thanks also to the people who allowed photographs they produced to enhance this book. For a complete list, see the Photo Credits pages near the end of the book.

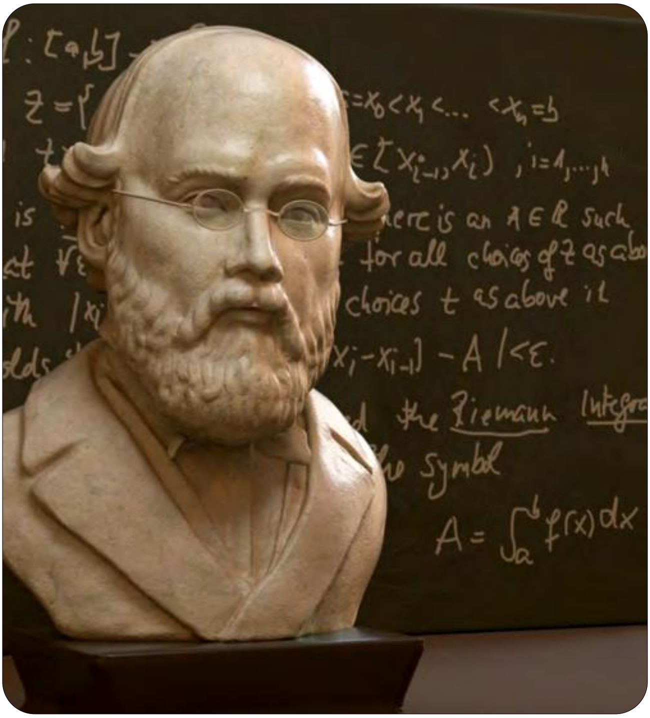
Special thanks to my wonderful partner Carrie Heeter, whose understanding and encouragement enabled me to work intensely on this book. Our cat Moon, whose picture is on page 44, helped provide relaxing breaks.

Sheldon Axler

Chapter 1
Riemann Integration

This brief chapter reviews Riemann integration. Riemann integration uses rectangles to approximate areas under graphs. This chapter begins by carefully presenting the definitions leading to the Riemann integral. The big result in the first section states that a continuous real-valued function on a closed bounded interval is Riemann integrable. The proof depends upon the theorem that continuous functions on closed bounded intervals are uniformly continuous.

The second section of this chapter focuses on several deficiencies of Riemann integration. As we will see, Riemann integration does not do everything we would like an integral to do. These deficiencies provide motivation in future chapters for the development of measures and integration with respect to measures.



Digital sculpture of Bernhard Riemann (1826–1866), whose method of integration is taught in calculus courses.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_1

1A Review: Riemann Integral

We begin with a few definitions needed before we can define the Riemann integral. Let $\mathbf{R}$ denote the complete ordered field of real numbers.

1.1 Definition ***partition***

Suppose $a, b \in \mathbf{R}$ with $a < b$. A *partition* of $[a, b]$ is a finite list of the form $x_0, x_1, \dots, x_n$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We use a partition $x_0, x_1, \dots, x_n$ of $[a, b]$ to think of $[a, b]$ as a union of closed subintervals, as follows:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n].$$

The next definition introduces clean notation for the infimum and supremum of the values of a function on some subset of its domain.

1.2 Definition ***notation for infimum and supremum of a function***

If f is a real-valued function and A is a subset of the domain of f, then

$$\inf_A f = \inf\{f(x) : x \in A\} \quad \text{and} \quad \sup_A f = \sup\{f(x) : x \in A\}.$$

The lower and upper Riemann sums, which we now define, approximate the area under the graph of a nonnegative function (or, more generally, the signed area corresponding to a real-valued function).

1.3 Definition ***lower and upper Riemann sums***

Suppose $f \colon [a, b] \to \mathbf{R}$ is a bounded function and P is a partition $x_0, \dots, x_n$ of $[a, b]$. The *lower Riemann sum* $L(f, P, [a, b])$ and the *upper Riemann sum* $U(f, P, [a, b])$ are defined by

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

and

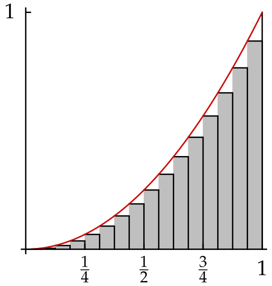
$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

Our intuition suggests that for a partition with only a small gap between consecutive points, the lower Riemann sum should be a bit less than the area under the graph, and the upper Riemann sum should be a bit more than the area under the graph.

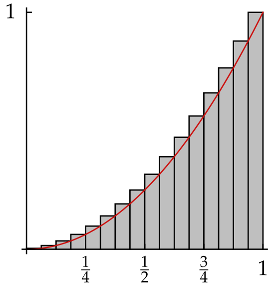
The pictures in the next example help convey the idea of these approximations. The base of the jth rectangle has length $x_j - x_{j-1}$ and has height $\inf\limits_{[x_{j-1}, x_j]} f$ for the lower Riemann sum and height $\sup\limits_{[x_{j-1}, x_j]} f$ for the upper Riemann sum.

1.4 **Example** *lower and upper Riemann sums*

Define $f\colon [0,1] \to \mathbf{R}$ by $f(x) = x^2$. Let P_n denote the partition $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$ of $[0,1]$.



The two figures here show the graph of f in red. The infimum of this function f is attained at the left endpoint of each subinterval $[\frac{j-1}{n}, \frac{j}{n}]$; the supremum is attained at the right endpoint.



$L(x^2, P_{16}, [0,1])$ is the sum of the areas of these rectangles.

$U(x^2, P_{16}, [0,1])$ is the sum of the areas of these rectangles.

For the partition P_n, we have $x_j - x_{j-1} = \frac{1}{n}$ for each $j = 1, \dots, n$. Thus

$$L(x^2, P_n, [0,1]) = \frac{1}{n} \sum_{j=1}^{n} \frac{(j-1)^2}{n^2} = \frac{2n^2 - 3n + 1}{6n^2}$$

and

$$U(x^2, P_n, [0,1]) = \frac{1}{n} \sum_{j=1}^{n} \frac{j^2}{n^2} = \frac{2n^2 + 3n + 1}{6n^2},$$

as you should verify [use the formula $1 + 4 + 9 + \cdots + n^2 = \frac{n(2n^2+3n+1)}{6}$].

The next result states that adjoining more points to a partition increases the lower Riemann sum and decreases the upper Riemann sum.

1.5 ***inequalities with Riemann sums***

Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function and P, P' are partitions of $[a,b]$ such that the list defining P is a sublist of the list defining P'. Then

$$L(f, P, [a,b]) \le L(f, P', [a,b]) \le U(f, P', [a,b]) \le U(f, P, [a,b]).$$

Proof To prove the first inequality, suppose P is the partition $x_0, \dots, x_n$ and P' is the partition $x'_0, \dots, x'_N$ of $[a,b]$. For each $j = 1, \dots, n$, there exist $k \in \{0, \dots, N-1\}$ and a positive integer m such that $x_{j-1} = x'_k < x'_{k+1} < \cdots < x'_{k+m} = x_j$. We have

$$
\begin{aligned}
(x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f &= \sum_{i=1}^{m} (x'_{k+i} - x'_{k+i-1}) \inf_{[x_{j-1}, x_j]} f \\
&\le \sum_{i=1}^{m} (x'_{k+i} - x'_{k+i-1}) \inf_{[x'_{k+i-1}, x'_{k+i}]} f.
\end{aligned}
$$

The inequality above implies that $L(f, P, [a, b]) \le L(f, P', [a, b])$.

The middle inequality in this result follows from the observation that the infimum of each set of real numbers is less than or equal to the supremum of that set.

The proof of the last inequality in this result is similar to the proof of the first inequality and is left to the reader. ■

The following result states that if the function is fixed, then each lower Riemann sum is less than or equal to each upper Riemann sum.

1.6 ***lower Riemann sums $\le$ upper Riemann sums***

Suppose $f: [a, b] \to \mathbf{R}$ is a bounded function and P, P' are partitions of $[a, b]$. Then

$$L(f, P, [a, b]) \le U(f, P', [a, b]).$$

Proof Let P'' be the partition of $[a, b]$ obtained by merging the lists that define P and P'. Then

$$
\begin{aligned}
L(f, P, [a, b]) &\le L(f, P'', [a, b]) \\
&\le U(f, P'', [a, b]) \\
&\le U(f, P', [a, b]),
\end{aligned}
$$

where all three inequalities above come from 1.5. ■

We have been working with lower and upper Riemann sums. Now we define the lower and upper Riemann integrals.

1.7 Definition ***lower and upper Riemann integrals***

Suppose $f: [a, b] \to \mathbf{R}$ is a bounded function. The *lower Riemann integral* $L(f, [a, b])$ and the *upper Riemann integral* $U(f, [a, b])$ of f are defined by

$$L(f, [a, b]) = \sup_{P} L(f, P, [a, b])$$

and

$$U(f, [a, b]) = \inf_{P} U(f, P, [a, b]),$$

where the supremum and infimum above are taken over all partitions P of $[a, b]$.

In the definition above, we take the supremum (over all partitions) of the lower Riemann sums because adjoining more points to a partition increases the lower Riemann sum (by 1.5) and should provide a more accurate estimate of the area under the graph. Similarly, in the definition above, we take the infimum (over all partitions) of the upper Riemann sums because adjoining more points to a partition decreases the upper Riemann sum (by 1.5) and should provide a more accurate estimate of the area under the graph.

Our first result about the lower and upper Riemann integrals is an easy inequality.

1.8 ***lower Riemann integral $\leq$ upper Riemann integral***

Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. Then

$$L(f,[a,b]) \leq U(f,[a,b]).$$

Proof The desired inequality follows from the definitions and 1.6. ■

The lower Riemann integral and the upper Riemann integral can both be reasonably considered to be the area under the graph of a function. Which one should we use? The pictures in Example 1.4 suggest that these two quantities are the same for the function in that example; we will soon verify this suspicion. However, as we will see in the next section, there are functions for which the lower Riemann integral does not equal the upper Riemann integral.

Instead of choosing between the lower Riemann integral and the upper Riemann integral, the standard procedure in Riemann integration is to consider only functions for which those two quantities are equal. This decision has the huge advantage of making the Riemann integral behave as we wish with respect to the sum of two functions (see Exercise 4 in this section).

1.9 Definition ***Riemann integrable; Riemann integral***

- A bounded function on a closed bounded interval is called *Riemann integrable* if its lower Riemann integral equals its upper Riemann integral.
- If $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable, then the *Riemann integral* $\int_a^b f$ is defined by
$$\int_a^b f = L(f,[a,b]) = U(f,[a,b]).$$

Let $\mathbf{Z}$ denote the set of integers and $\mathbf{Z}^+$ denote the set of positive integers.

1.10 Example *computing a Riemann integral*

Define $f\colon [0,1] \to \mathbf{R}$ by $f(x) = x^2$. Then

$$U(f,[0,1]) \leq \inf_{n \in \mathbf{Z}^+} \frac{2n^2+3n+1}{6n^2} = \frac{1}{3} = \sup_{n \in \mathbf{Z}^+} \frac{2n^2-3n+1}{6n^2} \leq L(f,[0,1]),$$

where the two inequalities above come from Example 1.4 and the two equalities easily follow from dividing the numerators and denominators of both fractions above by n^2.

The paragraph above shows that $U(f,[0,1]) \leq \frac{1}{3} \leq L(f,[0,1])$. When combined with 1.8, this shows that $L(f,[0,1]) = U(f,[0,1]) = \frac{1}{3}$. Thus f is Riemann integrable and

$$\int_0^1 f = \frac{1}{3}.$$

Our definition of Riemann integration is actually a small modification of Riemann's definition that was proposed by Gaston Darboux (1842–1917).

Now we come to a key result regarding Riemann integration. Uniform continuity provides the major tool that makes the proof work.

1.11 *continuous functions are Riemann integrable*

Every continuous real-valued function on each closed bounded interval is Riemann integrable.

Proof Suppose $a, b \in \mathbf{R}$ with $a < b$ and $f\colon [a,b] \to \mathbf{R}$ is a continuous function (thus by a standard theorem from undergraduate real analysis, f is bounded and is uniformly continuous). Let $\varepsilon > 0$. Because f is uniformly continuous, there exists $\delta > 0$ such that

1.12 $$|f(s) - f(t)| < \varepsilon \text{ for all } s, t \in [a,b] \text{ with } |s - t| < \delta.$$

Let $n \in \mathbf{Z}^+$ be such that $\frac{b-a}{n} < \delta$.

Let P be the equally spaced partition $a = x_0, x_1, \ldots, x_n = b$ of $[a,b]$ with

$$x_j - x_{j-1} = \frac{b-a}{n}$$

for each $j = 1, \ldots, n$. Then

$$\begin{aligned} U(f,[a,b]) - L(f,[a,b]) &\leq U(f,P,[a,b]) - L(f,P,[a,b]) \\ &= \frac{b-a}{n} \sum_{j=1}^{n} \Big(\sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f \Big) \\ &\leq (b-a)\varepsilon, \end{aligned}$$

where the first equality follows from the definitions of $U(f,[a,b])$ and $L(f,[a,b])$ and the last inequality follows from 1.12.

We have shown that $U(f,[a,b]) - L(f,[a,b]) \leq (b-a)\varepsilon$ for all $\varepsilon > 0$. Thus 1.8 implies that $L(f,[a,b]) = U(f,[a,b])$. Hence f is Riemann integrable. ■

An alternative notation for $\int_a^b f$ is $\int_a^b f(x)\,dx$. Here x is a dummy variable, so we could also write $\int_a^b f(t)\,dt$ or use another variable. This notation becomes useful when we want to write something like $\int_0^1 x^2\,dx$ instead of using function notation.

The next result gives a frequently used estimate for a Riemann integral.

1.13 *bounds on Riemann integral*

Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Then

$$(b-a) \inf_{[a,b]} f \leq \int_a^b f \leq (b-a) \sup_{[a,b]} f$$

Proof Let P be the trivial partition $a = x_0, x_1 = b$. Then

$$(b-a) \inf_{[a,b]} f = L(f, P, [a,b]) \leq L(f, [a,b]) = \int_a^b f,$$

proving the first inequality in the result.

The second inequality in the result is proved similarly and is left to the reader. ■

EXERCISES 1A

1 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function such that

$$L(f, P, [a,b]) = U(f, P, [a,b])$$

for some partition P of $[a,b]$. Prove that f is a constant function on $[a,b]$.

2 Suppose $a \leq s < t \leq b$. Define $f\colon [a,b] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } s < x < t, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Riemann integrable on $[a,b]$ and that $\int_a^b f = t - s$.

3 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. Prove that f is Riemann integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f, P, [a,b]) - L(f, P, [a,b]) < \varepsilon.$$

4 Suppose $f, g\colon [a,b] \to \mathbf{R}$ are Riemann integrable. Prove that $f + g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

5 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that the function $-f$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (-f) = -\int_a^b f.$$

6 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Suppose $g\colon [a,b] \to \mathbf{R}$ is a function such that $g(x) = f(x)$ for all except finitely many $x \in [a,b]$. Prove that g is Riemann integrable on $[a,b]$ and

$$\int_a^b g = \int_a^b f.$$

7 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. For $n \in \mathbf{Z}^+$, let P_n denote the partition that divides $[a,b]$ into 2^n intervals of equal size. Prove that

$$L(f,[a,b]) = \lim_{n\to\infty} L(f,P_n,[a,b]) \text{ and } U(f,[a,b]) = \lim_{n\to\infty} U(f,P_n,[a,b]).$$

8 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f\bigl(a + \tfrac{j(b-a)}{n}\bigr).$$

9 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that if $c,d \in \mathbf{R}$ and $a \le c < d \le b$, then f is Riemann integrable on $[c,d]$.
[*To say that f is Riemann integrable on $[c,d]$ means that f with its domain restricted to $[c,d]$ is Riemann integrable.*]

10 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function and $c \in (a,b)$. Prove that f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on $[a,c]$ and f is Riemann integrable on $[c,b]$. Furthermore, prove that if these conditions hold, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

11 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Define $F\colon [a,b] \to \mathbf{R}$ by

$$F(t) = \begin{cases} 0 & \text{if } t = a, \\ \displaystyle\int_a^t f & \text{if } t \in (a,b]. \end{cases}$$

Prove that F is continuous on $[a,b]$.

12 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that $|f|$ is Riemann integrable and that

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

13 Suppose $f\colon [a,b] \to \mathbf{R}$ is an increasing function, meaning that $c,d \in [a,b]$ with $c < d$ implies $f(c) \le f(d)$. Prove that f is Riemann integrable on $[a,b]$.

14 Suppose $f_1, f_2, \ldots$ is a sequence of Riemann integrable functions on $[a,b]$ such that $f_1, f_2, \ldots$ converges uniformly on $[a,b]$ to a function $f\colon [a,b] \to \mathbf{R}$. Prove that f is Riemann integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

1B Riemann Integral Is Not Good Enough

The Riemann integral works well enough to be taught to millions of calculus students around the world each year. However, the Riemann integral has several deficiencies. In this section, we discuss the following three issues:

- Riemann integration does not handle functions with many discontinuities;
- Riemann integration does not handle unbounded functions;
- Riemann integration does not work well with limits.

In Chapter 2, we will start to construct a theory to remedy these problems.

We begin with the following example of a function that is not Riemann integrable.

1.14 Example *a function that is not Riemann integrable*

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

If $[a,b] \subset [0,1]$ with $a < b$, then

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} f = 1$$

because $[a,b]$ contains an irrational number and contains a rational number. Thus $L(f,P,[0,1]) = 0$ and $U(f,P,[0,1]) = 1$ for every partition P of $[0,1]$. Hence $L(f,[0,1]) = 0$ and $U(f,[0,1]) = 1$. Because $L(f,[0,1]) \neq U(f,[0,1])$, we conclude that f is not Riemann integrable.

This example is disturbing because (as we will see later), there are far fewer rational numbers than irrational numbers. Thus f should, in some sense, have integral 0. However, the Riemann integral of f is not defined.

Trying to apply the definition of the Riemann integral to unbounded functions would lead to undesirable results, as shown in the next example.

1.15 Example *Riemann integration does not work with unbounded functions*

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

If $x_0, x_1, \ldots, x_n$ is a partition of $[0,1]$, then $\sup\limits_{[x_0, x_1]} f = \infty$. Thus if we tried to apply the definition of the upper Riemann sum to f, we would have $U(f,P,[0,1]) = \infty$ for every partition P of $[0,1]$.

However, we should consider the area under the graph of f to be 2, not ∞, because

$$\lim_{a \downarrow 0} \int_a^1 f = \lim_{a \downarrow 0} (2 - 2\sqrt{a}) = 2.$$

Calculus courses deal with the previous example by defining $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ to be $\lim_{a\downarrow 0}\int_a^1 \frac{1}{\sqrt{x}}\,dx$. If using this approach and

$$f(x) = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}},$$

then we would define $\int_0^1 f$ to be

$$\lim_{a\downarrow 0}\int_a^{1/2} f + \lim_{b\uparrow 1}\int_{1/2}^b f.$$

However, the idea of taking Riemann integrals over subdomains and then taking limits can fail with more complicated functions, as shown in the next example.

1.16 Example *area seems to make sense, but Riemann integral is not defined*

Let $r_1, r_2, \ldots$ be a sequence that includes each rational number in $(0,1)$ exactly once and that includes no other numbers. For $k \in \mathbf{Z}^+$, define $f_k\colon [0,1] \to \mathbf{R}$ by

$$f_k(x) = \begin{cases} \frac{1}{\sqrt{x-r_k}} & \text{if } x > r_k, \\ 0 & \text{if } x \le r_k. \end{cases}$$

Define $f\colon [0,1] \to [0,\infty]$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{2^k}.$$

Because every nonempty open subinterval of $[0,1]$ contains a rational number, the function f is unbounded on every such subinterval. Thus the Riemann integral of f is undefined on every subinterval of $[0,1]$ with more than one element.

However, the area under the graph of each f_k is less than 2. The formula defining f then shows that we should expect the area under the graph of f to be less than 2 rather than undefined.

The next example shows that the pointwise limit of a sequence of Riemann integrable functions bounded by 1 need not be Riemann integrable.

1.17 Example *Riemann integration does not work well with pointwise limits*

Let $r_1, r_2, \ldots$ be a sequence that includes each rational number in $[0,1]$ exactly once and that includes no other numbers. For $k \in \mathbf{Z}^+$, define $f_k\colon [0,1] \to \mathbf{R}$ by

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \ldots, r_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_k is Riemann integrable and $\int_0^1 f_k = 0$, as you should verify.

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Clearly

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{for each } x \in [0,1].$$

However, f is not Riemann integrable (see Example 1.14) even though f is the pointwise limit of a sequence of integrable functions bounded by 1.

Because analysis relies heavily upon limits, a good theory of integration should allow for interchange of limits and integrals, at least when the functions are appropriately bounded. Thus the previous example points out a serious deficiency in Riemann integration.

Now we come to a positive result, but as we will see, even this result indicates that Riemann integration has some problems.

1.18 *interchanging Riemann integral and limit*

Suppose $a, b, M \in \mathbf{R}$ with $a < b$. Suppose $f_1, f_2, \dots$ is a sequence of Riemann integrable functions on $[a,b]$ such that

$$|f_k(x)| \le M$$

for all $k \in \mathbf{Z}^+$ and all $x \in [a,b]$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in [a,b]$. Define $f\colon [a,b] \to \mathbf{R}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

If f is Riemann integrable on $[a,b]$, then

$$\int_a^b f = \lim_{k\to\infty} \int_a^b f_k.$$

The result above suffers from two problems. The first problem is the undesirable hypothesis that the limit function f is Riemann integrable. Ideally, that property would follow from the other hypotheses, but Example 1.17 shows that this need not be true.

The difficulty in finding a simple Riemann-integration-based proof of the result above suggests that Riemann integration is not the ideal theory of integration.

The second problem with the result above is that its proof seems to be more intricate than the proofs of other results involving Riemann integration. We do not give a proof here of the result above. A clean proof of a stronger result is given in Chapter 3, using the tools of measure theory that we develop starting with the next chapter.

EXERCISES 1B

1 Define $f\colon [0,1] \to \mathbf{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational,} \\ \frac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive integer} \\ & \text{such that } a = \frac{m}{n} \text{ for some integer } m. \end{cases}$$

Show that f is Riemann integrable and compute $\int_0^1 f$.

2 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. Prove that f is Riemann integrable if and only if

$$L(-f,[a,b]) = -L(f,[a,b]).$$

3 Suppose $f,g\colon [a,b] \to \mathbf{R}$ are bounded functions. Prove that

$$L(f,[a,b]) + L(g,[a,b]) \le L(f+g,[a,b])$$

and

$$U(f+g,[a,b]) \le U(f,[a,b]) + U(g,[a,b]).$$

4 Give an example of bounded functions $f,g\colon [0,1] \to \mathbf{R}$ such that

$$L(f,[0,1]) + L(g,[0,1]) < L(f+g,[0,1])$$

and

$$U(f+g,[0,1]) < U(f,[0,1]) + U(g,[0,1]).$$

5 Give an example of a sequence of continuous real-valued functions $f_1, f_2, \ldots$ on $[0,1]$ and a continuous real-valued function f on $[0,1]$ such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for each $x \in [0,1]$ but

$$\int_0^1 f \ne \lim_{k\to\infty} \int_0^1 f_k.$$

Chapter 2
Measures

The last section of the previous chapter discusses several deficiencies of Riemann integration. To remedy those deficiencies, in this chapter we extend the notion of the length of an interval to a larger collection of subsets of **R**. This leads us to measures and then in the next chapter to integration with respect to measures.

We begin this chapter by investigating outer measure, which looks promising but fails to have a crucial property. That failure leads us to σ-algebras and measurable spaces. Then we define measures in an abstract context that can be applied to settings more general than **R**. Next, we construct Lebesgue measure on **R** as our desired extension of the notion of the length of an interval.



Fifth-century AD Roman ceiling mosaic in what is now a UNESCO World Heritage site in Ravenna, Italy. Giuseppe Vitali, who in 1905 proved result 2.18 in this chapter, was born and grew up in Ravenna, where perhaps he saw this mosaic. Could the memory of the translation-invariant feature of this mosaic have suggested to Vitali the translation invariance that is the heart of his proof of 2.18?

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S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_2

2A Outer Measure on $\mathbf{R}$

Motivation and Definition of Outer Measure

The Riemann integral arises from approximating the area under the graph of a function by sums of the areas of approximating rectangles. These rectangles have heights that approximate the values of the function on subintervals of the function's domain. The width of each approximating rectangle is the length of the corresponding subinterval. This length is the term $x_j - x_{j-1}$ in the definitions of the lower and upper Riemann sums (see 1.3).

To extend integration to a larger class of functions than the Riemann integrable functions, we will write the domain of a function as the union of subsets more complicated than the subintervals used in Riemann integration. We will need to assign a size to each of those subsets, where the size is an extension of the length of intervals.

For example, we expect the size of the set $(1,3) \cup (7,10)$ to be 5 (because the first interval has length 2, the second interval has length 3, and $2 + 3 = 5$).

Assigning a size to subsets of $\mathbf{R}$ that are more complicated than unions of open intervals becomes a nontrivial task. This chapter focuses on that task and its extension to other contexts. In the next chapter, we will see how to use the ideas developed in this chapter to create a rich theory of integration.

We begin by giving the expected definition of the length of an open interval, along with a notation for that length.

2.1 Definition ***length of open interval;*** $\ell(I)$

The *length* $\ell(I)$ of an open interval I is defined by

$$\ell(I) = \begin{cases} b - a & \text{if } I = (a,b) \text{ for some } a, b \in \mathbf{R} \text{ with } a < b, \\ 0 & \text{if } I = \varnothing, \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbf{R}, \\ \infty & \text{if } I = (-\infty, \infty). \end{cases}$$

Suppose $A \subset \mathbf{R}$. The size of A should be at most the sum of the lengths of a sequence of open intervals whose union contains A. Taking the infimum of all such sums gives a reasonable definition of the size of A, denoted $|A|$ and called the outer measure of A.

2.2 Definition ***outer measure;*** $|A|$

The *outer measure* $|A|$ of a set $A \subset \mathbf{R}$ is defined by

$$|A| = \inf\Bigl\{\sum_{k=1}^{\infty} \ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \bigcup_{k=1}^{\infty} I_k\Bigr\}.$$

The definition of outer measure involves an infinite sum. The infinite sum $\sum_{k=1}^{\infty} t_k$ of a sequence $t_1, t_2, \ldots$ of elements of $[0, \infty]$ is defined to be ∞ if some $t_k = \infty$. Otherwise, $\sum_{k=1}^{\infty} t_k$ is defined to be the limit (possibly ∞) of the increasing sequence $t_1, t_1 + t_2, t_1 + t_2 + t_3, \ldots$ of partial sums; thus

$$\sum_{k=1}^{\infty} t_k = \lim_{n \to \infty} \sum_{k=1}^{n} t_k.$$

2.3 **Example** *finite sets have outer measure* 0

Suppose $A = \{a_1, \ldots, a_n\}$ is a finite set of real numbers. Suppose $\varepsilon > 0$. Define a sequence $I_1, I_2, \ldots$ of open intervals by

$$I_k = \begin{cases} (a_k - \varepsilon, a_k + \varepsilon) & \text{if } k \leq n, \\ \varnothing & \text{if } k > n. \end{cases}$$

Then $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains A. Clearly $\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon n$. Hence $|A| \leq 2\varepsilon n$. Because ε is an arbitrary positive number, this implies that $|A| = 0$.

Good Properties of Outer Measure

Outer measure has several nice properties that are discussed in this subsection. We begin with a result that improves upon the example above.

2.4 ***countable sets have outer measure*** 0

Every countable subset of **R** has outer measure 0.

Proof Suppose $A = \{a_1, a_2, \ldots\}$ is a countable subset of **R**. Let $\varepsilon > 0$. For $k \in \mathbf{Z}^+$, let

$$I_k = \Big(a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}\Big).$$

Then $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains A. Because

$$\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon,$$

we have $|A| \leq 2\varepsilon$. Because ε is an arbitrary positive number, this implies that $|A| = 0$. ∎

The result above, along with the result that the set **Q** of rational numbers is countable, implies that **Q** has outer measure 0. We will soon show that there are far fewer rational numbers than real numbers (see 2.17). Thus the equation $|\mathbf{Q}| = 0$ indicates that outer measure has a good property that we want any reasonable notion of size to possess.

The next result shows that outer measure does the right thing with respect to set inclusion.

2.5 *outer measure preserves order*

Suppose A and B are subsets of $\mathbf{R}$ with $A \subset B$. Then $|A| \leq |B|$.

Proof Suppose $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains B. Then the union of this sequence of open intervals also contains A. Hence

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k).$$

Taking the infimum over all sequences of open intervals whose union contains B, we have $|A| \leq |B|$. ■

We expect that the size of a subset of $\mathbf{R}$ should not change if the set is shifted to the right or to the left. The next definition allows us to be more precise.

2.6 Definition *translation;* $t + A$

If $t \in \mathbf{R}$ and $A \subset \mathbf{R}$, then the *translation* $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

If $t > 0$, then $t + A$ is obtained by moving the set A to the right t units on the real line; if $t < 0$, then $t + A$ is obtained by moving the set A to the left $|t|$ units.

Translation does not change the length of an open interval. Specifically, if $t \in \mathbf{R}$ and $a, b \in [-\infty, \infty]$, then $t + (a, b) = (t + a, t + b)$ and thus $\ell\big(t + (a, b)\big) = \ell\big((a, b)\big)$. Here we are using the standard convention that $t + (-\infty) = -\infty$ and $t + \infty = \infty$.

The next result states that translation invariance carries over to outer measure.

2.7 *outer measure is translation invariant*

Suppose $t \in \mathbf{R}$ and $A \subset \mathbf{R}$. Then $|t + A| = |A|$.

Proof Suppose $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains A. Then $t + I_1, t + I_2, \ldots$ is a sequence of open intervals whose union contains $t + A$. Thus

$$|t + A| \leq \sum_{k=1}^{\infty} \ell(t + I_k) = \sum_{k=1}^{\infty} \ell(I_k).$$

Taking the infimum of the last term over all sequences $I_1, I_2, \ldots$ of open intervals whose union contains A, we have $|t + A| \leq |A|$.

To get the inequality in the other direction, note that $A = -t + (t + A)$. Thus applying the inequality from the previous paragraph, with A replaced by $t + A$ and t replaced by $-t$, we have $|A| = |-t + (t + A)| \leq |t + A|$. Hence $|t + A| = |A|$. ■

The union of the intervals $(1,4)$ and $(3,5)$ is the interval $(1,5)$. Thus

$$\ell\big((1,4)\cup(3,5)\big) < \ell\big((1,4)\big) + \ell\big((3,5)\big)$$

because the left side of the inequality above equals 4 and the right side equals 5. The direction of the inequality above is explained by noting that the interval $(3,4)$, which is the intersection of $(1,4)$ and $(3,5)$, has its length counted twice on the right side of the inequality above.

The example of the paragraph above should provide intuition for the direction of the inequality in the next result. The property of satisfying the inequality in the result below is called *countable subadditivity* because it applies to sequences of subsets.

2.8 *countable subadditivity of outer measure*

Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbf{R}$. Then

$$\Big|\bigcup_{k=1}^{\infty} A_k\Big| \leq \sum_{k=1}^{\infty} |A_k|.$$

Proof If $|A_k| = \infty$ for some $k \in \mathbf{Z}^+$, then the inequality above clearly holds. Thus assume $|A_k| < \infty$ for all $k \in \mathbf{Z}^+$.

Let $\varepsilon > 0$. For each $k \in \mathbf{Z}^+$, let $I_{1,k}, I_{2,k}, \ldots$ be a sequence of open intervals whose union contains A_k such that

$$\sum_{j=1}^{\infty} \ell(I_{j,k}) \leq \frac{\varepsilon}{2^k} + |A_k|.$$

Thus

2.9 $$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{j,k}) \leq \varepsilon + \sum_{k=1}^{\infty} |A_k|.$$

The doubly indexed collection of open intervals $\{I_{j,k} : j,k \in \mathbf{Z}^+\}$ can be rearranged into a sequence of open intervals whose union contains $\bigcup_{k=1}^{\infty} A_k$ as follows, where in step k (start with $k = 2$, then $k = 3,4,5,\ldots$) we adjoin the $k-1$ intervals whose indices add up to k:

$$\underbrace{I_{1,1}}_{2},\ \underbrace{I_{1,2}, I_{2,1}}_{3},\ \underbrace{I_{1,3}, I_{2,2}, I_{3,1}}_{4},\ \underbrace{I_{1,4}, I_{2,3}, I_{3,2}, I_{4,1}}_{5},\ \underbrace{I_{1,5}, I_{2,4}, I_{3,3}, I_{4,2}, I_{5,1}}_{\text{sum of the two indices is 6}}, \ldots .$$

Inequality 2.9 shows that the sum of the lengths of the intervals listed above is less than or equal to $\varepsilon + \sum_{k=1}^{\infty} |A_k|$. Thus $\big|\bigcup_{k=1}^{\infty} A_k\big| \leq \varepsilon + \sum_{k=1}^{\infty} |A_k|$. Because ε is an arbitrary positive number, this implies that $\big|\bigcup_{k=1}^{\infty} A_k\big| \leq \sum_{k=1}^{\infty} |A_k|$. ■

Countable subadditivity implies finite subadditivity, meaning that

$$|A_1 \cup \cdots \cup A_n| \leq |A_1| + \cdots + |A_n|$$

for all $A_1, \ldots, A_n \subset \mathbf{R}$, because we can take $A_k = \varnothing$ for $k > n$ in 2.8.

The countable subadditivity of outer measure, as proved above, adds to our list of nice properties enjoyed by outer measure.

Outer Measure of Closed Bounded Interval

One more good property of outer measure that we should prove is that if $a < b$, then the outer measure of the closed interval $[a,b]$ is $b-a$. Indeed, if $\varepsilon > 0$, then $(a-\varepsilon, b+\varepsilon), \varnothing, \varnothing, \ldots$ is a sequence of open intervals whose union contains $[a,b]$. Thus $|[a,b]| \leq b-a+2\varepsilon$. Because this inequality holds for all $\varepsilon > 0$, we conclude that

$$|[a,b]| \leq b-a.$$

Is the inequality in the other direction obviously true to you? If so, think again, because a proof of the inequality in the other direction requires that the completeness of $\mathbf{R}$ is used in some form. For example, suppose $\mathbf{R}$ was a countable set (which is not true, as we will soon see, but the uncountability of $\mathbf{R}$ is not obvious). Then we would have $|[a,b]| = 0$ (by 2.4). Thus something deeper than you might suspect is going on with the ingredients needed to prove that $|[a,b]| \geq b-a$.

The following definition will be useful when we prove that $|[a,b]| \geq b-a$.

2.10 Definition *open cover*

Suppose $A \subset \mathbf{R}$.

- A collection $\mathcal{C}$ of open subsets of $\mathbf{R}$ is called an *open cover* of A if A is contained in the union of all the sets in $\mathcal{C}$.
- An open cover $\mathcal{C}$ of A is said to have a *finite subcover* if A is contained in the union of some finite list of sets in $\mathcal{C}$.

2.11 Example *open covers and finite subcovers*

- The collection $\{(k, k+2) : k \in \mathbf{Z}^+\}$ is an open cover of $[2,5]$ because $[2,5] \subset \bigcup_{k=1}^{\infty}(k, k+2)$. This open cover has a finite subcover because $[2,5] \subset (1,3) \cup (2,4) \cup (3,5) \cup (4,6)$.
- The collection $\{(k, k+2) : k \in \mathbf{Z}^+\}$ is an open cover of $[2,\infty)$ because $[2,\infty] \subset \bigcup_{k=1}^{\infty}(k, k+2)$. This open cover does not have a finite subcover because there do not exist finitely many sets of the form $(k, k+2)$ whose union contains $[2,\infty)$.
- The collection $\{(0, 2-\frac{1}{k}) : k \in \mathbf{Z}^+\}$ is an open cover of $(1,2)$ because $(1,2) \subset \bigcup_{k=1}^{\infty}(0, 2-\frac{1}{k})$. This open cover does not have a finite subcover because there do not exist finitely many sets of the form $(0, 2-\frac{1}{k})$ whose union contains $(1,2)$.

The next result will be our major tool in the proof that $|[a,b]| \geq b-a$. Although we need only the result as stated, be sure to see Exercise 4 in this section, which when combined with the next result gives a characterization of the closed bounded subsets of $\mathbf{R}$. Note that the following proof uses the completeness property of the real numbers (by asserting that the supremum of a certain nonempty bounded set exists).

2.12 *Heine–Borel Theorem*

Every open cover of a closed bounded subset of $\mathbf{R}$ has a finite subcover.

To provide visual clues, we usually denote closed sets by F and open sets by G.

Proof Suppose F is a closed bounded subset of $\mathbf{R}$ and $\mathcal{C}$ is an open cover of F.

First consider the case where $F = [a, b]$ for some $a, b \in \mathbf{R}$ with $a < b$. Thus $\mathcal{C}$ is an open cover of $[a, b]$. Let

$$D = \{d \in [a, b] : [a, d] \text{ has a finite subcover from } \mathcal{C}\}.$$

Note that $a \in D$ (because $a \in G$ for some $G \in \mathcal{C}$). Thus D is not the empty set. Let

$$s = \sup D.$$

Thus $s \in [a, b]$. Hence there exists an open set $G \in \mathcal{C}$ such that $s \in G$. Let $\delta > 0$ be such that $(s - \delta, s + \delta) \subset G$. Because $s = \sup D$, there exist $d \in (s - \delta, s]$ and $n \in \mathbf{Z}^+$ and $G_1, \dots, G_n \in \mathcal{C}$ such that

$$[a, d] \subset G_1 \cup \cdots \cup G_n.$$

Now

2.13 $$[a, d'] \subset G \cup G_1 \cup \cdots \cup G_n$$

for all $d' \in [s, s + \delta)$.Thus $d' \in D$ for all $d' \in [s, s + \delta) \cap [a, b]$. This implies that $s = b$. Furthermore, 2.13 with $d' = b$ shows that $[a, b]$ has a finite subcover from $\mathcal{C}$, completing the proof in the case where $F = [a, b]$.

Now suppose F is an arbitrary closed bounded subset of $\mathbf{R}$ and that $\mathcal{C}$ is an open cover of F. Let $a, b \in \mathbf{R}$ be such that $F \subset [a, b]$. Now $\mathcal{C} \cup \{\mathbf{R} \setminus F\}$ is an open cover of $\mathbf{R}$ and hence is an open cover of $[a, b]$ (here $\mathbf{R} \setminus F$ denotes the set complement of F in $\mathbf{R}$). By our first case, there exist $G_1, \dots, G_n \in \mathcal{C}$ such that

$$[a, b] \subset G_1 \cup \cdots \cup G_n \cup (\mathbf{R} \setminus F).$$

Thus

$$F \subset G_1 \cup \cdots \cup G_n,$$

completing the proof. ■



Saint-Affrique, the small town in southern France where Émile Borel (1871–1956) was born. Borel first stated and proved what we call the Heine–Borel Theorem in 1895. Earlier, Eduard Heine (1821–1881) and others had used similar results.

Now we can prove that closed intervals have the expected outer measure.

2.14 ***outer measure of a closed interval***

Suppose $a, b \in \mathbf{R}$, with $a < b$. Then $|[a, b]| = b - a$.

Proof See the first paragraph of this subsection for the proof that $|[a, b]| \leq b - a$.

To prove the inequality in the other direction, suppose $I_1, I_2, \ldots$ is a sequence of open intervals such that $[a, b] \subset \bigcup_{k=1}^{\infty} I_k$. By the Heine–Borel Theorem (2.12), there exists $n \in \mathbf{Z}^+$ such that

2.15 $$[a, b] \subset I_1 \cup \cdots \cup I_n.$$

We will now prove by induction on n that the inclusion above implies that

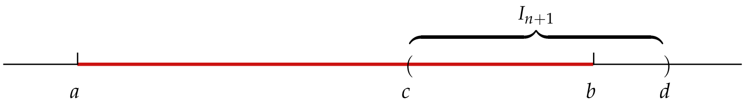
2.16 $$\sum_{k=1}^{n} \ell(I_k) \geq b - a.$$

This will then imply that $\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) \geq b - a$, completing the proof that $|[a, b]| \geq b - a$.

To get started with our induction, note that 2.15 clearly implies 2.16 if $n = 1$. Now for the induction step: Suppose $n > 1$ and 2.15 implies 2.16 for all choices of $a, b \in \mathbf{R}$ with $a < b$. Suppose $I_1, \ldots, I_n, I_{n+1}$ are open intervals such that

$$[a, b] \subset I_1 \cup \cdots \cup I_n \cup I_{n+1}.$$

Thus b is in at least one of the intervals $I_1, \ldots, I_n, I_{n+1}$. By relabeling, we can assume that $b \in I_{n+1}$. Suppose $I_{n+1} = (c, d)$. If $c \leq a$, then $\ell(I_{n+1}) \geq b - a$ and there is nothing further to prove; thus we can assume that $a < c < b < d$, as shown in the figure below.



Hence

$$[a, c] \subset I_1 \cup \cdots \cup I_n.$$

By our induction hypothesis, we have $\sum_{k=1}^{n} \ell(I_k) \geq c - a$. Thus

$$\begin{aligned} \sum_{k=1}^{n+1} \ell(I_k) &\geq (c - a) + \ell(I_{n+1}) \\ &= (c - a) + (d - c) \\ &= d - a \\ &\geq b - a, \end{aligned}$$

completing the proof. ■

Alice was beginning to get very tired of sitting by her sister on the bank, and of having nothing to do: once or twice she had peeped into the book her sister was reading, but it had no pictures or conversations in it, "and what is the use of a book," thought Alice, "without pictures or conversation?"
– opening paragraph of Alice's Adventures in Wonderland, *by Lewis Carroll*

The result above easily implies that the outer measure of each open interval equals its length (see Exercise 6).

The previous result has the following important corollary. You may be familiar with Georg Cantor's (1845–1918) original proof of the next result. The proof using outer measure that is presented here gives an interesting alternative to Cantor's proof.

2.17 *nontrivial intervals are uncountable*

Every interval in **R** that contains at least two distinct elements is uncountable.

Proof Suppose I is an interval that contains $a, b \in \mathbf{R}$ with $a < b$. Then

$$|I| \geq |[a,b]| = b - a > 0,$$

where the first inequality above holds because outer measure preserves order (see 2.5) and the equality above comes from 2.14. Because every countable subset of **R** has outer measure 0 (see 2.4), we can conclude that I is uncountable. ■

Outer Measure is Not Additive

We have had several results giving nice properties of outer measure. Now we come to an unpleasant property of outer measure.

*Outer measure led to the proof above that **R** is uncountable. This application of outer measure to prove a result that seems unconnected with outer measure is an indication that outer measure has serious mathematical value.*

If outer measure were a perfect way to assign a size as an extension of the lengths of intervals, then the outer measure of the union of two disjoint sets would equal the sum of the outer measures of the two sets. Sadly, the next result states that outer measure does not have this property.

In the next section, we begin the process of getting around the next result, which will lead us to measure theory.

2.18 *nonadditivity of outer measure*

There exist disjoint subsets A and B of **R** such that

$$|A \cup B| \neq |A| + |B|.$$

Proof For $a \in [-1, 1]$, let $\tilde{a}$ be the set of numbers in $[-1, 1]$ that differ from a by a rational number. In other words,

$$\tilde{a} = \{c \in [-1,1] : a - c \in \mathbf{Q}\}.$$

Think of $\tilde{a}$ as the equivalence class of a under the equivalence relation that declares $a, c \in [-1,1]$ to be equivalent if $a - c \in \mathbf{Q}$.

If $a, b \in [-1, 1]$ and $\tilde{a} \cap \tilde{b} \neq \varnothing$, then $\tilde{a} = \tilde{b}$. (Proof: Suppose there exists $d \in \tilde{a} \cap \tilde{b}$. Then $a - d$ and $b - d$ are rational numbers; subtracting, we conclude that $a - b$ is a rational number. The equation $a - c = (a - b) + (b - c)$ now implies that if $c \in [-1, 1]$, then $a - c$ is a rational number if and only if $b - c$ is a rational number. In other words, $\tilde{a} = \tilde{b}$.)

Clearly $a \in \tilde{a}$ for each $a \in [-1,1]$. Thus $[-1,1] = \bigcup_{a\in[-1,1]} \tilde{a}$.

Let V be a set that contains exactly one element in each of the distinct sets in

$$\{\tilde{a} : a \in [-1,1]\}.$$

In other words, for every $a \in [-1,1]$, the set $V \cap \tilde{a}$ has exactly one element.

This step involves the Axiom of Choice, as discussed after this proof. The set V arises by choosing one element from each equivalence class.

Let $r_1, r_2, \ldots$ be a sequence of distinct rational numbers such that

$$[-2,2] \cap \mathbf{Q} = \{r_1, r_2, \ldots\}.$$

Then

$$[-1,1] \subset \bigcup_{k=1}^{\infty} (r_k + V),$$

where the set inclusion above holds because if $a \in [-1,1]$, then letting v be the unique element of $V \cap \tilde{a}$, we have $a - v \in \mathbf{Q}$, which implies that $a = r_k + v \in r_k + V$ for some $k \in \mathbf{Z}^+$.

The set inclusion above, the order-preserving property of outer measure (2.5), and the countable subadditivity of outer measure (2.8) imply

$$|[-1,1]| \leq \sum_{k=1}^{\infty} |r_k + V|.$$

We know that $|[-1,1]| = 2$ (from 2.14). The translation invariance of outer measure (2.7) thus allows us to rewrite the inequality above as

$$2 \leq \sum_{k=1}^{\infty} |V|.$$

Thus $|V| > 0$.

Note that the sets $r_1 + V, r_2 + V, \ldots$ are disjoint. (Proof: Suppose there exists $t \in (r_j + V) \cap (r_k + V)$. Then $t = r_j + v_1 = r_k + v_2$ for some $v_1, v_2 \in V$, which implies that $v_1 - v_2 = r_k - r_j \in \mathbf{Q}$. Our construction of V now implies that $v_1 = v_2$, which implies that $r_j = r_k$, which implies that $j = k$.)

Let $n \in \mathbf{Z}^+$. Clearly

$$\bigcup_{k=1}^{n} (r_k + V) \subset [-3,3]$$

because $V \subset [-1,1]$ and each $r_k \in [-2,2]$. The set inclusion above implies that

2.19 $$\Big|\bigcup_{k=1}^{n} (r_k + V)\Big| \leq 6.$$

However

2.20 $$\sum_{k=1}^{n} |r_k + V| = \sum_{k=1}^{n} |V| = n\,|V|.$$

Now 2.19 and 2.20 suggest that we choose $n \in \mathbf{Z}^+$ such that $n|V| > 6$. Thus

2.21
$$\Bigl|\bigcup_{k=1}^{n}(r_k+V)\Bigr| < \sum_{k=1}^{n}|r_k+V|.$$

If we had $|A\cup B| = |A|+|B|$ for all disjoint subsets A, B of $\mathbf{R}$, then by induction on n we would have $\Bigl|\bigcup_{k=1}^{n} A_k\Bigr| = \sum_{k=1}^{n}|A_k|$ for all disjoint subsets $A_1,\dots,A_n$ of $\mathbf{R}$. However, 2.21 tells us that no such result holds. Thus there exist disjoint subsets A, B of $\mathbf{R}$ such that $|A\cup B| \neq |A|+|B|$. ■

The Axiom of Choice, which belongs to set theory, states that if $\mathcal{E}$ is a set whose elements are disjoint nonempty sets, then there exists a set D that contains exactly one element in each set that is an element of $\mathcal{E}$. We used the Axiom of Choice to construct the set D that was used in the last proof.

A small minority of mathematicians objects to the use of the Axiom of Choice. Thus we will keep track of where we need to use it. Even if you do not like to use the Axiom of Choice, the previous result warns us away from trying to prove that outer measure is additive (any such proof would need to contradict the Axiom of Choice, which is consistent with the standard axioms of set theory).

EXERCISES 2A

1 Prove that if A and B are subsets of $\mathbf{R}$ and $|B| = 0$, then $|A\cup B| = |A|$.

2 Suppose $A \subset \mathbf{R}$ and $t \in \mathbf{R}$. Let $tA = \{ta : a \in A\}$. Prove that $|tA| = |t|\,|A|$. [*Assume that* $0\cdot\infty$ *is defined to be* 0.]

3 Prove that if $A, B \subset \mathbf{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

4 Suppose F is a subset of $\mathbf{R}$ with the property that every open cover of F has a finite subcover. Prove that F is closed and bounded.

5 Suppose $\mathcal{A}$ is a set of closed subsets of $\mathbf{R}$ such that $\bigcap_{F\in\mathcal{A}} F = \varnothing$. Prove that if $\mathcal{A}$ contains at least one bounded set, then there exist $n \in \mathbf{Z}^+$ and $F_1,\dots,F_n \in \mathcal{A}$ such that $F_1\cap\cdots\cap F_n = \varnothing$.

6 Prove that if $a, b \in \mathbf{R}$ and $a < b$, then

$$|(a,b)| = |[a,b)| = |(a,b]| = b-a.$$

7 Suppose a, b, c, d are real numbers with $a < b$ and $c < d$. Prove that

$$|(a,b)\cup(c,d)| = (b-a)+(d-c) \text{ if and only if } (a,b)\cap(c,d) = \varnothing.$$

8 Prove that if $A \subset \mathbf{R}$ and $t > 0$, then $|A| = |A\cap(-t,t)| + \bigl|A\cap\bigl(\mathbf{R}\setminus(-t,t)\bigr)\bigr|$.

9 Prove that $|A| = \lim_{t\to\infty}|A\cap(-t,t)|$ for all $A \subset \mathbf{R}$.

10 Prove that $|[0,1] \setminus \mathbf{Q}| = 1$.

11 Prove that if $I_1, I_2, \ldots$ is a disjoint sequence of open intervals, then

$$\Big|\bigcup_{k=1}^{\infty} I_k\Big| = \sum_{k=1}^{\infty} \ell(I_k).$$

12 Suppose $r_1, r_2, \ldots$ is a sequence that contains every rational number. Let

$$F = \mathbf{R} \setminus \bigcup_{k=1}^{\infty} \Big(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k}\Big).$$

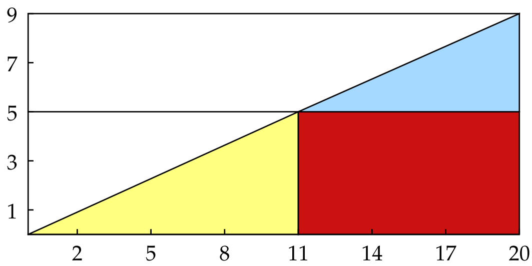
(a) Show that F is a closed subset of $\mathbf{R}$.

(b) Prove that if I is an interval contained in F, then I contains at most one element.

(c) Prove that $|F| = \infty$.

13 Suppose $\varepsilon > 0$. Prove that there exists a subset F of $[0,1]$ such that F is closed, every element of F is an irrational number, and $|F| > 1 - \varepsilon$.

14 Consider the following figure, which is drawn accurately to scale.



(a) Show that the right triangle whose vertices are $(0,0)$, $(20,0)$, and $(20,9)$ has area 90.

[*We have not defined area yet, but just use the elementary formulas for the areas of triangles and rectangles that you learned long ago.*]

(b) Show that the yellow (lower) right triangle has area 27.5.

(c) Show that the red rectangle has area 45.

(d) Show that the blue (upper) right triangle has area 18.

(e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.

(f) Seeing the figure above, most people expect parts (a) and (e) to have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ.

[*You may be tempted to think that what we have here is a two-dimensional example similar to the result about the nonadditivity of outer measure* (2.18). *However, genuine examples of nonadditivity require much more complicated sets than in this example.*]

2B Measurable Spaces and Functions

The last result in the previous section showed that outer measure is not additive. Could this disappointing result perhaps be fixed by using some notion other than outer measure for the size of a subset of **R**? The next result answers this question by showing that there does not exist a notion of size, called the Greek letter mu (μ) in the result below, that has all the desirable properties.

Property (c) in the result below is called *countable additivity*. Countable additivity is a highly desirable property because we want to be able to prove theorems about limits (the heart of analysis!), which requires countable additivity.

2.22 *nonexistence of extension of length to all subsets of* R

There does not exist a function μ with all the following properties:

(a) μ is a function from the set of subsets of **R** to $[0, \infty]$.

(b) $\mu(I) = \ell(I)$ for every open interval I of **R**.

(c) $\mu\Bigl(\bigcup_{k=1}^{\infty} A_k\Bigr) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of **R**.

(d) $\mu(t + A) = \mu(A)$ for every $A \subset \mathbf{R}$ and every $t \in \mathbf{R}$.

We will show that μ has all the properties of outer measure that were used in the proof of 2.18.

Proof Suppose there exists a function μ with all the properties listed in the statement of this result.

Observe that $\mu(\varnothing) = 0$, as follows from (b) because the empty set is an open interval with length 0.

If $A \subset B \subset \mathbf{R}$, then $\mu(A) \leq \mu(B)$, as follows from (c) because we can write B as the union of the disjoint sequence $A, B \setminus A, \varnothing, \varnothing, \ldots$; thus

$$\mu(B) = \mu(A) + \mu(B \setminus A) + 0 + 0 + \cdots = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

If $a, b \in \mathbf{R}$ with $a < b$, then $(a, b) \subset [a, b] \subset (a - \varepsilon, b + \varepsilon)$ for every $\varepsilon > 0$. Thus $b - a \leq \mu([a, b]) \leq b - a + 2\varepsilon$ for every $\varepsilon > 0$. Hence $\mu([a, b]) = b - a$.

If $A_1, A_2, \ldots$ is a sequence of subsets of **R**, then $A_1, A_2 \setminus A_1, A_3 \setminus (A_1 \cup A_2), \ldots$ is a disjoint sequence of subsets of **R** whose union is $\bigcup_{k=1}^{\infty} A_k$. Thus

$$\begin{aligned}
\mu\Bigl(\bigcup_{k=1}^{\infty} A_k\Bigr) &= \mu\bigl(A_1 \cup (A_2 \setminus A_1) \cup \bigl(A_3 \setminus (A_1 \cup A_2)\bigr) \cup \cdots\bigr) \\
&= \mu(A_1) + \mu(A_2 \setminus A_1) + \mu\bigl(A_3 \setminus (A_1 \cup A_2)\bigr) + \cdots \\
&\leq \sum_{k=1}^{\infty} \mu(A_k),
\end{aligned}$$

where the second equality follows from the countable additivity of μ.

We have shown that μ has all the properties of outer measure that were used in the proof of 2.18. Repeating the proof of 2.18, we see that there exist disjoint subsets A, B of $\mathbf{R}$ such that $\mu(A \cup B) \neq \mu(A) + \mu(B)$. Thus the disjoint sequence $A, B, \emptyset, \emptyset, \ldots$ does not satisfy the countable additivity property required by (c). This contradiction completes the proof. ■

σ-Algebras

The last result shows that we need to give up one of the desirable properties in our goal of extending the notion of size from intervals to more general subsets of $\mathbf{R}$. We cannot give up 2.22(b) because the size of an interval needs to be its length. We cannot give up 2.22(c) because countable additivity is needed to prove theorems about limits. We cannot give up 2.22(d) because a size that is not translation invariant does not satisfy our intuitive notion of size as a generalization of length.

Thus we are forced to relax the requirement in 2.22(a) that the size is defined for all subsets of $\mathbf{R}$. Experience shows that to have a viable theory that allows for taking limits, the collection of subsets for which the size is defined should be closed under complementation and closed under countable unions. Thus we make the following definition.

2.23 Definition ***σ-algebra***

Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then $\mathcal{S}$ is called a *σ-algebra* on X if the following three conditions are satisfied:

- $\emptyset \in \mathcal{S}$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\displaystyle\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

Make sure you verify that the examples in all three bullet points below are indeed σ-algebras. The verification is obvious for the first two bullet points. For the third bullet point, you need to use the result that the countable union of countable sets is countable (see the proof of 2.8 for an example of how a doubly indexed list can be converted to a singly indexed sequence). The exercises contain some additional examples of σ-algebras.

2.24 Example *σ-algebras*

- Suppose X is a set. Then clearly $\{\emptyset, X\}$ is a σ-algebra on X.
- Suppose X is a set. Then clearly the set of all subsets of X is a σ-algebra on X.
- Suppose X is a set. Then the set of all subsets E of X such that E is countable or $X \setminus E$ is countable is a σ-algebra on X.

Now we come to some easy but important properties of σ-algebras.

2.25 *σ-algebras are closed under countable intersection*

Suppose $\mathcal{S}$ is a σ-algebra on a set X. Then

(a) $X \in \mathcal{S}$;

(b) if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$;

(c) if $E_1, E_2, \dots$ is a sequence of elements of $\mathcal{S}$, then $\displaystyle\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$.

Proof Because $\varnothing \in \mathcal{S}$ and $X = X \setminus \varnothing$, the first two bullet points in the definition of σ-algebra (2.23) imply that $X \in \mathcal{S}$, proving (a).

Suppose $D, E \in \mathcal{S}$. Then $D \cup E$ is the union of the sequence $D, E, \varnothing, \varnothing, \dots$ of elements of $\mathcal{S}$. Thus the third bullet point in the definition of σ-algebra (2.23) implies that $D \cup E \in \mathcal{S}$.

De Morgan's Laws tell us that

$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E).$$

If $D, E \in \mathcal{S}$, then the right side of the equation above is in $\mathcal{S}$; hence $X \setminus (D \cap E) \in \mathcal{S}$; thus the complement in X of $X \setminus (D \cap E)$ is in $\mathcal{S}$; in other words, $D \cap E \in \mathcal{S}$.

Because $D \setminus E = D \cap (X \setminus E)$, we see that if $D, E \in \mathcal{S}$, then $D \setminus E \in \mathcal{S}$, completing the proof of (b).

Finally, suppose $E_1, E_2, \dots$ is a sequence of elements of $\mathcal{S}$. De Morgan's Laws tell us that

$$X \setminus \bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (X \setminus E_k).$$

The right side of the equation above is in $\mathcal{S}$. Hence the left side is in $\mathcal{S}$, which implies that $X \setminus (X \setminus \bigcap_{k=1}^{\infty} E_k) \in \mathcal{S}$. In other words, $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$, proving (c). ■

The word *measurable* is used in the terminology below because in the next section we introduce a size function, called a measure, defined on measurable sets.

2.26 Definition *measurable space; measurable set*

- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of $\mathcal{S}$ is called an *$\mathcal{S}$-measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context.

For example, if $X = \mathbf{R}$ and $\mathcal{S}$ is the set of all subsets of $\mathbf{R}$ that are countable or have a countable complement, then the set of rational numbers is $\mathcal{S}$-measurable but the set of positive real numbers is not $\mathcal{S}$-measurable.

Borel Subsets of $\mathbf{R}$

The next result guarantees that there is a smallest σ-algebra on a set X containing a given set $\mathcal{A}$ of subsets of X.

2.27 *smallest σ-algebra containing a collection of subsets*

Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X.

Proof There is at least one σ-algebra on X that contains $\mathcal{A}$ because the σ-algebra consisting of all subsets of X contains $\mathcal{A}$.

Let $\mathcal{S}$ be the intersection of all σ-algebras on X that contain $\mathcal{A}$. Then $\varnothing \in \mathcal{S}$ because $\varnothing$ is an element of each σ-algebra on X that contains $\mathcal{A}$.

Suppose $E \in \mathcal{S}$. Thus E is in every σ-algebra on X that contains $\mathcal{A}$. Thus $X \setminus E$ is in every σ-algebra on X that contains $\mathcal{A}$. Hence $X \setminus E \in \mathcal{S}$.

Suppose $E_1, E_2, \dots$ is a sequence of elements of $\mathcal{S}$. Thus each E_k is in every σ-algebra on X that contains $\mathcal{A}$. Thus $\bigcup_{k=1}^{\infty} E_k$ is in every σ-algebra on X that contains $\mathcal{A}$. Hence $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$, which completes the proof that $\mathcal{S}$ is a σ-algebra on X. ■

Using the terminology *smallest* for the intersection of all σ-algebras that contain a set $\mathcal{A}$ of subsets of X makes sense because the intersection of those σ-algebras is contained in every σ-algebra that contains $\mathcal{A}$.

2.28 Example *smallest σ-algebra*

- Suppose X is a set and $\mathcal{A}$ is the set of subsets of X that consist of exactly one element:
$$\mathcal{A} = \{\{x\} : x \in X\}.$$
Then the smallest σ-algebra on X containing $\mathcal{A}$ is the set of all subsets E of X such that E is countable or $X \setminus E$ is countable, as you should verify.

- Suppose $\mathcal{A} = \{(0,1), (0,\infty)\}$. Then the smallest σ-algebra on $\mathbf{R}$ containing $\mathcal{A}$ is $\{\varnothing, (0,1), (0,\infty), (-\infty,0] \cup [1,\infty), (-\infty,0], [1,\infty), (-\infty,1), \mathbf{R}\}$, as you should verify.

Now we come to a crucial definition.

2.29 Definition *Borel set*

The smallest σ-algebra on $\mathbf{R}$ containing all open subsets of $\mathbf{R}$ is called the collection of *Borel subsets* of $\mathbf{R}$. An element of this σ-algebra is called a *Borel set*.

We have defined the collection of Borel subsets of $\mathbf{R}$ to be the smallest σ-algebra on $\mathbf{R}$ containing all the open subsets of $\mathbf{R}$. We could have defined the collection of Borel subsets of $\mathbf{R}$ to be the smallest σ-algebra on $\mathbf{R}$ containing all the open intervals (because every open subset of $\mathbf{R}$ is the union of a sequence of open intervals).

2.30 **Example** *Borel sets*

- Every closed subset of $\mathbf{R}$ is a Borel set because every closed subset of $\mathbf{R}$ is the complement of an open subset of $\mathbf{R}$.
- Every countable subset of $\mathbf{R}$ is a Borel set because if $B = \{x_1, x_2, \ldots\}$, then $B = \bigcup_{k=1}^{\infty}\{x_k\}$, which is a Borel set because each $\{x_k\}$ is a closed set.
- Every half-open interval $[a, b)$ (where $a, b \in \mathbf{R}$) is a Borel set because $[a, b) = \bigcap_{k=1}^{\infty}(a - \frac{1}{k}, b)$.
- If $f\colon \mathbf{R} \to \mathbf{R}$ is a function, then the set of points at which f is continuous is the intersection of a sequence of open sets (see Exercise 12 in this section) and thus is a Borel set.

The intersection of every sequence of open subsets of $\mathbf{R}$ is a Borel set. However, the set of all such intersections is not the set of Borel sets (because it is not closed under countable unions). The set of all countable unions of countable intersections of open subsets of $\mathbf{R}$ is also not the set of Borel sets (because it is not closed under countable intersections). And so on *ad infinitum*—there is no finite procedure involving countable unions, countable intersections, and complements for constructing the collection of Borel sets.

We will see later that there exist subsets of $\mathbf{R}$ that are not Borel sets. However, any subset of $\mathbf{R}$ that you can write down in a concrete fashion is a Borel set.

Inverse Images

The next definition is used frequently in the rest of this chapter.

2.31 **Definition** *inverse image; $f^{-1}(A)$*

If $f\colon X \to Y$ is a function and $A \subset Y$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

2.32 **Example** *inverse images*

Suppose $f\colon [0, 4\pi] \to \mathbf{R}$ is defined by $f(x) = \sin x$. Then

$$\begin{aligned} f^{-1}\big((0, \infty)\big) &= (0, \pi) \cup (2\pi, 3\pi), \\ f^{-1}\big([0, 1]\big) &= [0, \pi] \cup [2\pi, 3\pi] \cup \{4\pi\}, \\ f^{-1}\big(\{-1\}\big) &= \{\tfrac{3\pi}{2}, \tfrac{7\pi}{2}\}, \\ f^{-1}\big((2, 3)\big) &= \varnothing, \end{aligned}$$

as you should verify.

Inverse images have good algebraic properties, as is shown in the next two results.

2.33 ***algebra of inverse images***

Suppose $f\colon X \to Y$ is a function. Then

(a) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ for every $A \subset Y$;

(b) $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of Y;

(c) $f^{-1}(\bigcap_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of Y.

Proof Suppose $A \subset Y$. For $x \in X$ we have

$$\begin{aligned} x \in f^{-1}(Y \setminus A) &\iff f(x) \in Y \setminus A \\ &\iff f(x) \notin A \\ &\iff x \notin f^{-1}(A) \\ &\iff x \in X \setminus f^{-1}(A). \end{aligned}$$

Thus $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, which proves (a).

To prove (b), suppose $\mathcal{A}$ is a set of subsets of Y. Then

$$\begin{aligned} x \in f^{-1}\Bigl(\bigcup_{A \in \mathcal{A}} A\Bigr) &\iff f(x) \in \bigcup_{A \in \mathcal{A}} A \\ &\iff f(x) \in A \text{ for some } A \in \mathcal{A} \\ &\iff x \in f^{-1}(A) \text{ for some } A \in \mathcal{A} \\ &\iff x \in \bigcup_{A \in \mathcal{A}} f^{-1}(A). \end{aligned}$$

Thus $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$, which proves (b).

Part (c) is proved in the same fashion as (b), with unions replaced by intersections and *for some* replaced by *for every*. ■

2.34 ***inverse image of a composition***

Suppose $f\colon X \to Y$ and $g\colon Y \to W$ are functions. Then

$$(g \circ f)^{-1}(A) = f^{-1}\bigl(g^{-1}(A)\bigr)$$

for every $A \subset W$.

Proof Suppose $A \subset W$. For $x \in X$ we have

$$\begin{aligned} x \in (g \circ f)^{-1}(A) \iff (g \circ f)(x) \in A &\iff g\bigl(f(x)\bigr) \in A \\ &\iff f(x) \in g^{-1}(A) \\ &\iff x \in f^{-1}\bigl(g^{-1}(A)\bigr). \end{aligned}$$

Thus $(g \circ f)^{-1}(A) = f^{-1}\bigl(g^{-1}(A)\bigr)$. ■

Measurable Functions

The next definition tells us which real-valued functions behave reasonably with respect to a σ-algebra on their domain.

2.35 Definition ***measurable function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f\colon X \to \mathbf{R}$ is called $\mathcal{S}$-*measurable* (or just *measurable* if $\mathcal{S}$ is clear from the context) if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset \mathbf{R}$.

2.36 Example *measurable functions*

- If $\mathcal{S} = \{\varnothing, X\}$, then the only $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ are the constant functions.
- If $\mathcal{S}$ is the set of all subsets of X, then every function from X to $\mathbf{R}$ is $\mathcal{S}$-measurable.
- If $\mathcal{S} = \{\varnothing, (-\infty, 0), [0, \infty), \mathbf{R}\}$ (which is a σ-algebra on $\mathbf{R}$), then a function $f\colon \mathbf{R} \to \mathbf{R}$ is $\mathcal{S}$-measurable if and only if f is constant on $(-\infty, 0)$ and f is constant on $[0, \infty)$.

Another class of examples comes from characteristic functions, which are defined below. The Greek letter chi (χ) is traditionally used to denote a characteristic function.

2.37 Definition ***characteristic function;*** χ_E

Suppose E is a subset of a set X. The *characteristic function* of E is the function $\chi_E\colon X \to \mathbf{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

The set X that contains E is not explicitly included in the notation χ_E because X will always be clear from the context.

2.38 Example *inverse image with respect to a characteristic function*

Suppose $(X, \mathcal{S})$ is a measurable space, $E \subset X$, and $B \subset \mathbf{R}$. Then

$$\chi_E{}^{-1}(B) = \begin{cases} E & \text{if } 0 \notin B \text{ and } 1 \in B, \\ X \setminus E & \text{if } 0 \in B \text{ and } 1 \notin B, \\ X & \text{if } 0 \in B \text{ and } 1 \in B, \\ \varnothing & \text{if } 0 \notin B \text{ and } 1 \notin B. \end{cases}$$

Thus we see that χ_E is an $\mathcal{S}$-measurable function if and only if $E \in \mathcal{S}$.

The definition of an $\mathcal{S}$-measurable function requires the inverse image of every Borel subset of $\mathbf{R}$ to be in $\mathcal{S}$. The next result shows that to verify that a function is $\mathcal{S}$-measurable, we can check the inverse images of a much smaller collection of subsets of $\mathbf{R}$.

Note that if $f\colon X \to \mathbf{R}$ is a function and $a \in \mathbf{R}$, then

$$f^{-1}\big((a,\infty)\big) = \{x \in X : f(x) > a\}.$$

2.39 *condition for measurable function*

Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to \mathbf{R}$ is a function such that

$$f^{-1}\big((a,\infty)\big) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an $\mathcal{S}$-measurable function.

Proof Let

$$\mathcal{T} = \{A \subset \mathbf{R} : f^{-1}(A) \in \mathcal{S}\}.$$

We want to show that every Borel subset of $\mathbf{R}$ is in $\mathcal{T}$. To do this, we will first show that $\mathcal{T}$ is a σ-algebra on $\mathbf{R}$.

Certainly $\varnothing \in \mathcal{T}$, because $f^{-1}(\varnothing) = \varnothing \in \mathcal{S}$.

If $A \in \mathcal{T}$, then $f^{-1}(A) \in \mathcal{S}$; hence

$$f^{-1}(\mathbf{R} \setminus A) = X \setminus f^{-1}(A) \in \mathcal{S}$$

by 2.33(a), and thus $\mathbf{R} \setminus \mathcal{A} \in \mathcal{T}$. In other words, $\mathcal{T}$ is closed under complementation.

If $A_1, A_2, \ldots \in \mathcal{T}$, then $f^{-1}(A_1), f^{-1}(A_2), \ldots \in \mathcal{S}$; hence

$$f^{-1}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \bigcup_{k=1}^{\infty} f^{-1}(A_k) \in \mathcal{S}$$

by 2.33(b), and thus $\bigcup_{k=1}^{\infty} A_k \in \mathcal{T}$. In other words, $\mathcal{T}$ is closed under countable unions. Thus $\mathcal{T}$ is a σ-algebra on $\mathbf{R}$.

By hypothesis, $\mathcal{T}$ contains $\{(a,\infty) : a \in \mathbf{R}\}$. Because $\mathcal{T}$ is closed under complementation, $\mathcal{T}$ also contains $\{(-\infty, b] : b \in \mathbf{R}\}$. Because the σ-algebra $\mathcal{T}$ is closed under finite intersections (by 2.25), we see that $\mathcal{T}$ contains $\{(a,b] : a, b \in \mathbf{R}\}$. Because $(a,b) = \bigcup_{k=1}^{\infty}(a, b - \frac{1}{k}]$ and $(-\infty, b) = \bigcup_{k=1}^{\infty}(-k, b - \frac{1}{k}]$ and $\mathcal{T}$ is closed under countable unions, we can conclude that $\mathcal{T}$ contains every open subset of $\mathbf{R}$.

Thus the σ-algebra $\mathcal{T}$ contains the smallest σ-algebra on $\mathbf{R}$ that contains all open subsets of $\mathbf{R}$. In other words, $\mathcal{T}$ contains every Borel subset of $\mathbf{R}$. Thus f is an $\mathcal{S}$-measurable function. ■

In the result above, we could replace the collection of sets $\{(a,\infty) : a \in \mathbf{R}\}$ by any collection of subsets of $\mathbf{R}$ such that the smallest σ-algebra containing that collection contains the Borel subsets of $\mathbf{R}$. For specific examples of such collections of subsets of $\mathbf{R}$, see Exercises 3–6.

We have been dealing with $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ in the context of an arbitrary set X and a σ-algebra $\mathcal{S}$ on X. An important special case of this setup is when X is a Borel subset of $\mathbf{R}$ and $\mathcal{S}$ is the set of Borel subsets of $\mathbf{R}$ that are contained in X (see Exercise 11 for another way of thinking about this σ-algebra). In this special case, the $\mathcal{S}$-measurable functions are called Borel measurable.

2.40 Definition ***Borel measurable function***

Suppose $X \subset \mathbf{R}$. A function $f\colon X \to \mathbf{R}$ is called *Borel measurable* if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbf{R}$.

If $X \subset \mathbf{R}$ and there exists a Borel measurable function $f\colon X \to \mathbf{R}$, then X must be a Borel set [because $X = f^{-1}(\mathbf{R})$].

If $X \subset \mathbf{R}$ and $f\colon X \to \mathbf{R}$ is a function, then f is a Borel measurable function if and only if $f^{-1}\big((a,\infty)\big)$ is a Borel set for every $a \in \mathbf{R}$ (use 2.39).

Suppose X is a set and $f\colon X \to \mathbf{R}$ is a function. The measurability of f depends upon the choice of a σ-algebra on X. If the σ-algebra is called $\mathcal{S}$, then we can discuss whether f is an $\mathcal{S}$-measurable function. If X is a Borel subset of $\mathbf{R}$, then $\mathcal{S}$ might be the set of Borel sets contained in X, in which case the phrase *Borel measurable* means the same as $\mathcal{S}$-measurable. However, whether or not $\mathcal{S}$ is a collection of Borel sets, we consider inverse images of Borel subsets of $\mathbf{R}$ when determining whether a function is $\mathcal{S}$-measurable.

The next result states that continuity interacts well with the notion of Borel measurability.

2.41 ***every continuous function is Borel measurable***

Every continuous real-valued function defined on a Borel subset of $\mathbf{R}$ is a Borel measurable function.

Proof Suppose $X \subset \mathbf{R}$ is a Borel set and $f\colon X \to \mathbf{R}$ is continuous. To prove that f is Borel measurable, fix $a \in \mathbf{R}$.

If $x \in X$ and $f(x) > a$, then (by the continuity of f) there exists $\delta_x > 0$ such that $f(y) > a$ for all $y \in (x - \delta_x, x + \delta_x) \cap X$. Thus

$$f^{-1}\big((a,\infty)\big) = \Bigg(\bigcup_{x \in f^{-1}((a,\infty))} (x - \delta_x, x + \delta_x)\Bigg) \cap X.$$

The union inside the large parentheses above is an open subset of $\mathbf{R}$; hence its intersection with X is a Borel set. Thus we can conclude that $f^{-1}\big((a,\infty)\big)$ is a Borel set.

Now 2.39 implies that f is a Borel measurable function. ■

Next we come to another class of Borel measurable functions. A similar definition could be made for decreasing functions, with a corresponding similar result.

2.42 Definition *increasing function*

Suppose $X \subset \mathbf{R}$ and $f \colon X \to \mathbf{R}$ is a function.

- f is called *increasing* if $f(x) \leq f(y)$ for all $x, y \in X$ with $x < y$.
- f is called *strictly increasing* if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

2.43 ***every increasing function is Borel measurable***

Every increasing function defined on a Borel subset of $\mathbf{R}$ is a Borel measurable function.

Proof Suppose $X \subset \mathbf{R}$ is a Borel set and $f \colon X \to \mathbf{R}$ is increasing. To prove that f is Borel measurable, fix $a \in \mathbf{R}$.

Let $b = \inf f^{-1}\big((a, \infty)\big)$. Then it is easy to see that

$$f^{-1}\big((a, \infty)\big) = (b, \infty) \cap X \quad \text{or} \quad f^{-1}\big((a, \infty)\big) = [b, \infty) \cap X.$$

Either way, we can conclude that $f^{-1}\big((a, \infty)\big)$ is a Borel set.

Now 2.39 implies that f is a Borel measurable function. ■

The next result shows that measurability interacts well with composition.

2.44 ***composition of measurable functions***

Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function. Suppose g is a real-valued Borel measurable function defined on a subset of $\mathbf{R}$ that includes the range of f. Then $g \circ f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function.

Proof Suppose $B \subset \mathbf{R}$ is a Borel set. Then (see 2.34)

$$(g \circ f)^{-1}(B) = f^{-1}\big(g^{-1}(B)\big).$$

Because g is a Borel measurable function, $g^{-1}(B)$ is a Borel subset of $\mathbf{R}$. Because f is an $\mathcal{S}$-measurable function, $f^{-1}\big(g^{-1}(B)\big) \in \mathcal{S}$. Thus the equation above implies that $(g \circ f)^{-1}(B) \in \mathcal{S}$. Thus $g \circ f$ is an $\mathcal{S}$-measurable function. ■

2.45 Example *if f is measurable, then so are $-f, \frac{1}{2}f, |f|, f^2$*

Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is $\mathcal{S}$-measurable. Then 2.44 implies that the functions $-f, \frac{1}{2}f, |f|, f^2$ are all $\mathcal{S}$-measurable functions because each of these functions can be written as the composition of f with a continuous (and thus Borel measurable) function g.

Specifically, take $g(x) = -x$, then $g(x) = \frac{1}{2}x$, then $g(x) = |x|$, and then $g(x) = x^2$.

Measurability also interacts well with algebraic operations, as shown in the next result.

2.46 *algebraic operations with measurable functions*

Suppose $(X, \mathcal{S})$ is a measurable space and $f, g \colon X \to \mathbf{R}$ are $\mathcal{S}$-measurable. Then

(a) $f + g$, $f - g$, and fg are $\mathcal{S}$-measurable functions;

(b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

Proof Suppose $a \in \mathbf{R}$. We will show that

2.47 $$(f+g)^{-1}\bigl((a,\infty)\bigr) = \bigcup_{r \in \mathbf{Q}} \Bigl(f^{-1}\bigl((r,\infty)\bigr) \cap g^{-1}\bigl((a-r,\infty)\bigr) \Bigr),$$

which implies that $(f+g)^{-1}\bigl((a,\infty)\bigr) \in \mathcal{S}$.

To prove 2.47, first suppose

$$x \in (f+g)^{-1}\bigl((a,\infty)\bigr).$$

Thus $a < f(x) + g(x)$. Hence the open interval $\bigl(a - g(x), f(x)\bigr)$ is nonempty, and thus it contains some rational number r. This implies that $r < f(x)$, which means that $x \in f^{-1}\bigl((r,\infty)\bigr)$, and $a - g(x) < r$, which implies that $x \in g^{-1}\bigl((a-r,\infty)\bigr)$. Thus x is an element of the right side of 2.47, completing the proof that the left side of 2.47 is contained in the right side.

The proof of the inclusion in the other direction is easier. Specifically, suppose $x \in f^{-1}\bigl((r,\infty)\bigr) \cap g^{-1}\bigl((a-r,\infty)\bigr)$ for some $r \in \mathbf{Q}$. Thus

$$r < f(x) \quad \text{and} \quad a - r < g(x).$$

Adding these two inequalities, we see that $a < f(x) + g(x)$. Thus x is an element of the left side of 2.47, completing the proof of 2.47. Hence $f + g$ is an $\mathcal{S}$-measurable function.

Example 2.45 tells us that $-g$ is an $\mathcal{S}$-measurable function. Thus $f - g$, which equals $f + (-g)$ is an $\mathcal{S}$-measurable function.

The easiest way to prove that fg is an $\mathcal{S}$-measurable function uses the equation

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

The operation of squaring an $\mathcal{S}$-measurable function produces an $\mathcal{S}$-measurable function (see Example 2.45), as does the operation of multiplication by $\frac{1}{2}$ (again, see Example 2.45). Thus the equation above implies that fg is an $\mathcal{S}$-measurable function, completing the proof of (a).

Suppose $g(x) \neq 0$ for all $x \in X$. The function defined on $\mathbf{R} \setminus \{0\}$ (a Borel subset of $\mathbf{R}$) that takes x to $\frac{1}{x}$ is continuous and thus is a Borel measurable function (by 2.41). Now 2.44 implies that $\frac{1}{g}$ is an $\mathcal{S}$-measurable function. Combining this result with what we have already proved about the product of $\mathcal{S}$-measurable functions, we conclude that $\frac{f}{g}$ is an $\mathcal{S}$-measurable function, proving (b). ■

The next result shows that the pointwise limit of a sequence of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable. This is a highly desirable property (recall that the set of Riemann integrable functions on some interval is not closed under taking pointwise limits; see Example 1.17).

2.48 *limit of $\mathcal{S}$-measurable functions*

Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in X$. Define $f \colon X \to \mathbf{R}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Then f is an $\mathcal{S}$-measurable function.

Proof Suppose $a \in \mathbf{R}$. We will show that

2.49
$$f^{-1}\big((a,\infty)\big) = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f_k^{-1}\big((a+\tfrac{1}{j},\infty)\big),$$

which implies that $f^{-1}\big((a,\infty)\big) \in \mathcal{S}$.

To prove 2.49, first suppose $x \in f^{-1}\big((a,\infty)\big)$. Thus there exists $j \in \mathbf{Z}^+$ such that $f(x) > a + \frac{1}{j}$. The definition of limit now implies that there exists $m \in \mathbf{Z}^+$ such that $f_k(x) > a + \frac{1}{j}$ for all $k \geq m$. Thus x is in the right side of 2.49, proving that the left side of 2.49 is contained in the right side.

To prove the inclusion in the other direction, suppose x is in the right side of 2.49. Thus there exist $j, m \in \mathbf{Z}^+$ such that $f_k(x) > a + \frac{1}{j}$ for all $k \geq m$. Taking the limit as $k \to \infty$, we see that $f(x) \geq a + \frac{1}{j} > a$. Thus x is in the left side of 2.49, completing the proof of 2.49. Thus f is an $\mathcal{S}$-measurable function. ∎

Occasionally we need to consider functions that take values in $[-\infty,\infty]$. For example, even if we start with a sequence of real-valued functions in 2.53, we might end up with functions with values in $[-\infty,\infty]$. Thus we extend the notion of Borel sets to subsets of $[-\infty,\infty]$, as follows.

2.50 Definition *Borel subsets of $[-\infty,\infty]$*

A subset of $[-\infty,\infty]$ is called a *Borel set* if its intersection with $\mathbf{R}$ is a Borel set.

In other words, a set $C \subset [-\infty,\infty]$ is a Borel set if and only if there exists a Borel set $B \subset \mathbf{R}$ such that $C = B$ or $C = B \cup \{\infty\}$ or $C = B \cup \{-\infty\}$ or $C = B \cup \{\infty, -\infty\}$.

You should verify that with the definition above, the set of Borel subsets of $[-\infty,\infty]$ is a σ-algebra on $[-\infty,\infty]$.

Next, we extend the definition of $\mathcal{S}$-measurable functions to functions taking values in $[-\infty,\infty]$.

2.51 Definition ***measurable function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f: X \to [-\infty, \infty]$ is called *$\mathcal{S}$-measurable* if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset [-\infty, \infty]$.

The next result, which is analogous to 2.39, states that we need not consider all Borel subsets of $[-\infty, \infty]$ when taking inverse images to determine whether or not a function with values in $[-\infty, \infty]$ is $\mathcal{S}$-measurable.

2.52 ***condition for measurable function***

Suppose $(X, \mathcal{S})$ is a measurable space and $f: X \to [-\infty, \infty]$ is a function such that

$$f^{-1}\big((a, \infty]\big) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an $\mathcal{S}$-measurable function.

The proof of the result above is left to the reader (also see Exercise 27 in this section).

We end this section by showing that the pointwise infimum and pointwise supremum of a sequence of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable.

2.53 ***infimum and supremum of a sequence of $\mathcal{S}$-measurable functions***

Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $[-\infty, \infty]$. Define $g, h: X \to [-\infty, \infty]$ by

$$g(x) = \inf\{f_k(x) : k \in \mathbf{Z}^+\} \quad \text{and} \quad h(x) = \sup\{f_k(x) : k \in \mathbf{Z}^+\}.$$

Then g and h are $\mathcal{S}$-measurable functions.

Proof Let $a \in \mathbf{R}$. The definition of the supremum implies that

$$h^{-1}\big((a, \infty]\big) = \bigcup_{k=1}^{\infty} f_k{}^{-1}\big((a, \infty]\big),$$

as you should verify. The equation above, along with 2.52, implies that h is an $\mathcal{S}$-measurable function.

Note that

$$g(x) = -\sup\{-f_k(x) : k \in \mathbf{Z}^+\}$$

for all $x \in X$. Thus the result about the supremum implies that g is an $\mathcal{S}$-measurable function. ∎

EXERCISES 2B

1 Show that $\mathcal{S} = \{\bigcup_{n \in K}(n, n+1] : K \subset \mathbf{Z}\}$ is a σ-algebra on $\mathbf{R}$.

2 Verify both bullet points in Example 2.28.

3 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, s] : r, s \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

4 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, n] : r \in \mathbf{Q}, n \in \mathbf{Z}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

5 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, r+1) : r \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

6 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{[r, \infty) : r \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

7 Prove that the collection of Borel subsets of $\mathbf{R}$ is translation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then $t + B$ is a Borel set.

8 Prove that the collection of Borel subsets of $\mathbf{R}$ is dilation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then tB (which is defined to be $\{tb : b \in B\}$) is a Borel set.

9 Give an example of a measurable space $(X, \mathcal{S})$ and a function $f \colon X \to \mathbf{R}$ such that $|f|$ is $\mathcal{S}$-measurable but f is not $\mathcal{S}$-measurable.

10 Show that the set of real numbers that have a decimal expansion with the digit 5 appearing infinitely often is a Borel set.

11 Suppose $\mathcal{T}$ is a σ-algebra on a set Y and $X \in \mathcal{T}$. Let $\mathcal{S} = \{E \in \mathcal{T} : E \subset X\}$.

(a) Show that $\mathcal{S} = \{F \cap X : F \in \mathcal{T}\}$.

(b) Show that $\mathcal{S}$ is a σ-algebra on X.

12 Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is a function.

(a) For $k \in \mathbf{Z}^+$, let

$$G_k = \{a \in \mathbf{R} : \text{there exists } \delta > 0 \text{ such that } |f(b) - f(c)| < \tfrac{1}{k} \\ \text{for all } b, c \in (a - \delta, a + \delta)\}.$$

Prove that G_k is an open subset of $\mathbf{R}$ for each $k \in \mathbf{Z}^+$.

(b) Prove that the set of points at which f is continuous equals $\bigcap_{k=1}^{\infty} G_k$.

(c) Conclude that the set of points at which f is continuous is a Borel set.

13 Suppose $(X, \mathcal{S})$ is a measurable space, $E_1, \dots, E_n$ are disjoint subsets of X, and $c_1, \dots, c_n$ are distinct nonzero real numbers. Prove that $c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n}$ is an $\mathcal{S}$-measurable function if and only if $E_1, \dots, E_n \in \mathcal{S}$.

14 (a) Suppose $f_1, f_2, \dots$ is a sequence of functions from a set X to $\mathbf{R}$. Explain why

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \dots \text{has a limit in } \mathbf{R}\}$$
$$= \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}\big((-\tfrac{1}{n}, \tfrac{1}{n})\big).$$

(b) Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Prove that

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \dots \text{has a limit in } \mathbf{R}\}$$

is an $\mathcal{S}$-measurable subset of X.

15 Suppose X is a set and $E_1, E_2, \dots$ is a disjoint sequence of subsets of X such that $\bigcup_{k=1}^{\infty} E_k = X$. Let $\mathcal{S} = \{\bigcup_{k \in K} E_k : K \subset \mathbf{Z}^+\}$.

(a) Show that $\mathcal{S}$ is a σ-algebra on X.

(b) Prove that a function from X to $\mathbf{R}$ is $\mathcal{S}$-measurable if and only if the function is constant on E_k for every $k \in \mathbf{Z}^+$.

16 Suppose $\mathcal{S}$ is a σ-algebra on a set X and $A \subset X$. Let

$$\mathcal{S}_A = \{E \in \mathcal{S} : A \subset E \text{ or } A \cap E = \varnothing\}.$$

(a) Prove that $\mathcal{S}_A$ is a σ-algebra on X.

(b) Suppose $f\colon X \to \mathbf{R}$ is a function. Prove that f is measurable with respect to $\mathcal{S}_A$ if and only if f is measurable with respect to $\mathcal{S}$ and f is constant on A.

17 Suppose X is a Borel subset of $\mathbf{R}$ and $f\colon X \to \mathbf{R}$ is a function such that $\{x \in X : f \text{ is not continuous at } x\}$ is a countable set. Prove f is a Borel measurable function.

18 Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable at every element of $\mathbf{R}$. Prove that f' is a Borel measurable function from $\mathbf{R}$ to $\mathbf{R}$.

19 Suppose X is a nonempty set and $\mathcal{S}$ is the σ-algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Give a characterization of the $\mathcal{S}$-measurable real-valued functions on X.

20 Suppose $(X, \mathcal{S})$ is a measurable space and $f, g\colon X \to \mathbf{R}$ are $\mathcal{S}$-measurable functions. Prove that if $f(x) > 0$ for all $x \in X$, then f^g (which is the function whose value at $x \in X$ equals $f(x)^{g(x)}$) is an $\mathcal{S}$-measurable function.

21 Prove 2.52.

22 Suppose $B \subset \mathbf{R}$ and $f\colon B \to \mathbf{R}$ is an increasing function. Prove that f is continuous at every element of B except for a countable subset of B.

23 Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a strictly increasing function. Prove that the inverse function $f^{-1}\colon f(\mathbf{R}) \to \mathbf{R}$ is a continuous function.
[*Note that this exercise does not have as a hypothesis that f is continuous.*]

24 Suppose $B \subset \mathbf{R}$ is a Borel set and $f\colon B \to \mathbf{R}$ is an increasing function. Prove that $f(B)$ is a Borel set.

25 Suppose $B \subset \mathbf{R}$ and $f\colon B \to \mathbf{R}$ is an increasing function. Prove that there exists a sequence $f_1, f_2, \ldots$ of strictly increasing functions from B to $\mathbf{R}$ such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for every $x \in B$.

26 Suppose $B \subset \mathbf{R}$ and $f\colon B \to \mathbf{R}$ is a bounded increasing function. Prove that there exists an increasing function $g\colon \mathbf{R} \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in B$.

27 Prove or give a counterexample: If $(X, \mathcal{S})$ is a measurable space and

$$f\colon X \to [-\infty, \infty]$$

is a function such that $f^{-1}\big((a, \infty)\big) \in \mathcal{S}$ for every $a \in \mathbf{R}$, then f is an $\mathcal{S}$-measurable function.

28 Suppose $f\colon B \to \mathbf{R}$ is a Borel measurable function. Define $g\colon \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B, \\ 0 & \text{if } x \in \mathbf{R} \setminus B. \end{cases}$$

Prove that g is a Borel measurable function.

29 Give an example of a measurable space $(X, \mathcal{S})$ and a family $\{f_t\}_{t\in\mathbf{R}}$ such that each f_t is an $\mathcal{S}$-measurable function from X to $[0,1]$, but the function $f\colon X \to [0,1]$ defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\}$$

is not $\mathcal{S}$-measurable.
[*Compare this exercise to 2.53, where the index set is $\mathbf{Z}^+$ rather than $\mathbf{R}$.*]

30 Show that

$$\lim_{j\to\infty}\Big(\lim_{k\to\infty}\big(\cos(j!\pi x)\big)^{2k}\Big) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

for every $x \in \mathbf{R}$.
[*This example is due to Henri Lebesgue.*]

2C Measures and Their Properties

Definition and Examples of Measures

The original motivation for the next definition came from trying to extend the notion of the length of an interval. However, the definition below allows us to discuss size in many more contexts. For example, we will see later that the area of a set in the plane or the volume of a set in higher dimensions fits into this structure. The word *measure* allows us to use a single word instead of repeating theorems for *length*, *area*, and *volume*.

2.54 Definition *measure*

Suppose X is a set and $\mathcal{S}$ is a σ-algebra on X. A *measure* on $(X, \mathcal{S})$ is a function $\mu\colon \mathcal{S} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every disjoint sequence $E_1, E_2, \dots$ of sets in $\mathcal{S}$.

The countable additivity that forms the key part of the definition above allows us to prove good limit theorems. Note that countable additivity implies finite additivity: if μ is a measure on $(X, \mathcal{S})$ and $E_1, \dots, E_n$ are disjoint sets in $\mathcal{S}$, then

$$\mu(E_1 \cup \cdots \cup E_n) = \mu(E_1) + \cdots + \mu(E_n),$$

as follows from applying the equation $\mu(\varnothing) = 0$ and countable additivity to the disjoint sequence $E_1, \dots, E_n, \varnothing, \varnothing, \dots$ of sets in $\mathcal{S}$.

In the mathematical literature, sometimes a measure on $(X, \mathcal{S})$ is just called a measure on X if the σ-algebra $\mathcal{S}$ is clear from the context.

The concept of a measure, as defined here, is sometimes called a positive measure *(although the phrase* nonnegative measure *would be more accurate).*

2.55 Example *measures*

- If X is a set, then *counting measure* is the measure μ defined on the σ-algebra of all subsets of X by setting $\mu(E) = n$ if E is a finite set containing exactly n elements and $\mu(E) = \infty$ if E is not a finite set.

- Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $c \in X$. Define the *Dirac* measure δ_c on $(X, \mathcal{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

  This measure is named in honor of mathematician and physicist Paul Dirac (1902–1984), who won the Nobel Prize for Physics in 1933 for his work combining relativity and quantum mechanics at the atomic level.

- Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $w\colon X \to [0,\infty]$ is a function. Define a measure μ on $(X,\mathcal{S})$ by
$$\mu(E) = \sum_{x \in E} w(x)$$
for $E \in \mathcal{S}$. [Here the sum is defined as the supremum of all finite subsums $\sum_{x \in D} w(x)$ as D ranges over all finite subsets of E.]

- Suppose X is a set and $\mathcal{S}$ is the σ-algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Define a measure μ on $(X,\mathcal{S})$ by
$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 3 & \text{if } E \text{ is uncountable.} \end{cases}$$

- Suppose $\mathcal{S}$ is the σ-algebra on $\mathbf{R}$ consisting of all subsets of $\mathbf{R}$. Then the function that takes a set $E \subset \mathbf{R}$ to $|E|$ (the outer measure of E) is not a measure because it is not finitely additive (see 2.18).

- Suppose $\mathcal{B}$ is the σ-algebra on $\mathbf{R}$ consisting of all Borel subsets of $\mathbf{R}$. We will see in the next section that outer measure is a measure on $(\mathbf{R},\mathcal{B})$.

The following terminology is frequently useful.

2.56 Definition ***measure space***

A *measure space* is an ordered triple $(X,\mathcal{S},\mu)$, where X is a set, $\mathcal{S}$ is a σ-algebra on X, and μ is a measure on $(X,\mathcal{S})$.

Properties of Measures

The hypothesis that $\mu(D) < \infty$ is needed in part (b) of the next result to avoid undefined expressions of the form $\infty - \infty$.

2.57 ***measure preserves order; measure of a set difference***

Suppose $(X,\mathcal{S},\mu)$ is a measure space and $D, E \in \mathcal{S}$ are such that $D \subset E$. Then

(a) $\mu(D) \leq \mu(E)$;

(b) $\mu(E \setminus D) = \mu(E) - \mu(D)$ provided that $\mu(D) < \infty$.

Proof Because $E = D \cup (E \setminus D)$ and this is a disjoint union, we have
$$\mu(E) = \mu(D) + \mu(E \setminus D) \geq \mu(D),$$
which proves (a). If $\mu(D) < \infty$, then subtracting $\mu(D)$ from both sides of the equation above proves (b). ∎

The countable additivity property of measures applies to disjoint countable unions. The following countable subadditivity property applies to countable unions that may not be disjoint unions.

2.58 ***countable subadditivity***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, E_2, \ldots \in \mathcal{S}$. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Proof Let $D_1 = \varnothing$ and $D_k = E_1 \cup \cdots \cup E_{k-1}$ for $k \geq 2$. Then

$$E_1 \setminus D_1, E_2 \setminus D_2, E_3 \setminus D_3, \ldots$$

is a disjoint sequence of subsets of X whose union equals $\bigcup_{k=1}^{\infty} E_k$. Thus

$$\begin{aligned} \mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) &= \mu\Big(\bigcup_{k=1}^{\infty} (E_k \setminus D_k)\Big) \\ &= \sum_{k=1}^{\infty} \mu(E_k \setminus D_k) \\ &\leq \sum_{k=1}^{\infty} \mu(E_k), \end{aligned}$$

where the second line above follows from the countable additivity of μ and the last line above follows from 2.57(a). ■

Note that countable subadditivity implies finite subadditivity: if μ is a measure on $(X, \mathcal{S})$ and $E_1, \ldots, E_n$ are sets in $\mathcal{S}$, then

$$\mu(E_1 \cup \cdots \cup E_n) \leq \mu(E_1) + \cdots + \mu(E_n),$$

as follows from applying the equation $\mu(\varnothing) = 0$ and countable subadditivity to the sequence $E_1, \ldots, E_n, \varnothing, \varnothing, \ldots$ of sets in $\mathcal{S}$.

The next result shows that measures behave well with increasing unions.

2.59 ***measure of an increasing union***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{S}$. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k \to \infty} \mu(E_k).$$

Proof If $\mu(E_k) = \infty$ for some $k \in \mathbf{Z}^+$, then the equation above holds because both sides equal ∞. Hence we can consider only the case where $\mu(E_k) < \infty$ for all $k \in \mathbf{Z}^+$.

For convenience, let $E_0 = \emptyset$. Then

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}),$$

where the union on the right side is a disjoint union. Thus

$$\begin{aligned}\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) &= \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1})\\ &= \lim_{k\to\infty} \sum_{j=1}^{k} \mu(E_j \setminus E_{j-1})\\ &= \lim_{k\to\infty} \sum_{j=1}^{k} \big(\mu(E_j) - \mu(E_{j-1})\big)\\ &= \lim_{k\to\infty} \mu(E_k),\end{aligned}$$



Another mew.

as desired. ■

Measures also behave well with respect to decreasing intersections (but see Exercise 10, which shows that the hypothesis $\mu(E_1) < \infty$ below cannot be deleted).

2.60 *measure of a decreasing intersection*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{S}$, with $\mu(E_1) < \infty$. Then

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k).$$

Proof One of De Morgan's Laws tells us that

$$E_1 \setminus \bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (E_1 \setminus E_k).$$

Now $E_1 \setminus E_1 \subset E_1 \setminus E_2 \subset E_1 \setminus E_3 \subset \cdots$ is an increasing sequence of sets in $\mathcal{S}$. Thus 2.59, applied to the equation above, implies that

$$\mu\Big(E_1 \setminus \bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_1 \setminus E_k).$$

Use 2.57(b) to rewrite the equation above as

$$\mu(E_1) - \mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \mu(E_1) - \lim_{k\to\infty} \mu(E_k),$$

which implies our desired result. ■

The next result is intuitively plausible—we expect that the measure of the union of two sets equals the measure of the first set plus the measure of the second set minus the measure of the set that has been counted twice.

2.61 *measure of a union*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $D, E \in \mathcal{S}$, with $\mu(D \cap E) < \infty$. Then

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E).$$

Proof We have

$$D \cup E = \big(D \setminus (D \cap E)\big) \cup \big(E \setminus (D \cap E)\big) \cup (D \cap E).$$

The right side of the equation above is a disjoint union. Thus

$$\begin{aligned}\mu(D \cup E) &= \mu\big(D \setminus (D \cap E)\big) + \mu\big(E \setminus (D \cap E)\big) + \mu(D \cap E)\\ &= \big(\mu(D) - \mu(D \cap E)\big) + \big(\mu(E) - \mu(D \cap E)\big) + \mu(D \cap E)\\ &= \mu(D) + \mu(E) - \mu(D \cap E),\end{aligned}$$

as desired. ■

EXERCISES 2C

1 Explain why there does not exist a measure space $(X, \mathcal{S}, \mu)$ with the property that $\{\mu(E) : E \in \mathcal{S}\} = [0, 1)$.

Let $2^{\mathbf{Z}^+}$ denote the σ-algebra on $\mathbf{Z}^+$ consisting of all subsets of $\mathbf{Z}^+$.

2 Suppose μ is a measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$. Prove that there is a sequence $w_1, w_2, \dots$ in $[0, \infty]$ such that

$$\mu(E) = \sum_{k \in E} w_k$$

for every set $E \subset \mathbf{Z}^+$.

3 Give an example of a measure μ on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ such that

$$\{\mu(E) : E \subset \mathbf{Z}^+\} = [0, 1].$$

4 Give an example of a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = \{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1].$$

5 Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$. Prove that if $\mathcal{A}$ is a set of disjoint sets in $\mathcal{S}$ such that $\mu(A) > 0$ for every $A \in \mathcal{A}$, then $\mathcal{A}$ is a countable set.

6 Find all $c \in [3, \infty)$ such that there exists a measure space $(X, \mathcal{S}, \mu)$ with

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,c].$$

7 Give an example of a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,\infty].$$

8 Give an example of a set X, a σ-algebra $\mathcal{S}$ of subsets of X, a set $\mathcal{A}$ of subsets of X such that the smallest σ-algebra on X containing $\mathcal{A}$ is $\mathcal{S}$, and two measures μ and ν on $(X, \mathcal{S})$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$ and $\mu(X) = \nu(X) < \infty$, but $\mu \neq \nu$.

9 Suppose μ and ν are measures on a measurable space $(X, \mathcal{S})$. Prove that $\mu + \nu$ is a measure on $(X, \mathcal{S})$. [Here $\mu + \nu$ is the usual sum of two functions: if $E \in \mathcal{S}$, then $(\mu + \nu)(E) = \mu(E) + \nu(E)$.]

10 Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$ such that

$$\mu\Bigl(\bigcap_{k=1}^{\infty} E_k\Bigr) \neq \lim_{k\to\infty} \mu(E_k).$$

11 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $C, D, E \in \mathcal{S}$ are such that

$$\mu(C \cap D) < \infty, \mu(C \cap E) < \infty, \text{ and } \mu(D \cap E) < \infty.$$

Find and prove a formula for $\mu(C \cup D \cup E)$ in terms of $\mu(C)$, $\mu(D)$, $\mu(E)$, $\mu(C \cap D)$, $\mu(C \cap E)$, $\mu(D \cap E)$, and $\mu(C \cap D \cap E)$.

12 Suppose X is a set and $\mathcal{S}$ is the σ-algebra of all subsets E of X such that E is countable or $X \setminus E$ is countable. Give a complete description of the set of all measures on $(X, \mathcal{S})$.

2D Lebesgue Measure

Additivity of Outer Measure on Borel Sets

Recall that there exist disjoint sets $A, B \in \mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$ (see 2.18). Thus outer measure, despite its name, is not a measure on the σ-algebra of all subsets of $\mathbf{R}$.

Our main goal in this section is to prove that outer measure, when restricted to the Borel subsets of $\mathbf{R}$, is a measure. Throughout this section, be careful about trying to simplify proofs by applying properties of measures to outer measure, even if those properties seem intuitively plausible. For example, there are subsets $A \subset B \subset \mathbf{R}$ with $|A| < \infty$ but $|B \setminus A| \neq |B| - |A|$ [compare to 2.57(b)].

The next result is our first step toward the goal of proving that outer measure restricted to the Borel sets is a measure.

2.62 ***additivity of outer measure if one of the sets is open***

Suppose A and G are disjoint subsets of $\mathbf{R}$ and G is open. Then

$$|A \cup G| = |A| + |G|.$$

Proof We can assume that $|G| < \infty$ because otherwise both $|A \cup G|$ and $|A| + |G|$ equal ∞.

Subadditivity (see 2.8) implies that $|A \cup G| \leq |A| + |G|$. Thus we need to prove the inequality only in the other direction.

First consider the case where $G = (a, b)$ for some $a, b \in \mathbf{R}$ with $a < b$. We can assume that $a, b \notin A$ (because changing a set by at most two points does not change its outer measure). Let $I_1, I_2, \ldots$ be a sequence of open intervals whose union contains $A \cup G$. For each $n \in \mathbf{Z}^+$, let

$$J_n = I_n \cap (-\infty, a), \quad K_n = I_n \cap (a, b), \quad L_n = I_n \cap (b, \infty).$$

Then

$$\ell(I_n) = \ell(J_n) + \ell(K_n) + \ell(L_n).$$

Now $J_1, L_1, J_2, L_2, \ldots$ is a sequence of open intervals whose union contains A and $K_1, K_2, \ldots$ is a sequence of open intervals whose union contains G. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} \ell(I_n) &= \sum_{n=1}^{\infty} \big(\ell(J_n) + \ell(L_n)\big) + \sum_{n=1}^{\infty} \ell(K_n)\\ &\geq |A| + |G|.\end{aligned}$$

The inequality above implies that $|A \cup G| \geq |A| + |G|$, completing the proof that $|A \cup G| = |A| + |G|$ in this special case.

Using induction on m, we can now conclude that if $m \in \mathbf{Z}^+$ and G is a union of m disjoint open intervals that are all disjoint from A, then $|A \cup G| = |A| + |G|$.

Now suppose G is an arbitrary open subset of $\mathbf{R}$ that is disjoint from A. Then $G = \bigcup_{n=1}^{\infty} I_n$ for some sequence of disjoint open intervals $I_1, I_2, \ldots$, each of which is disjoint from A. Now for each $m \in \mathbf{Z}^+$ we have

$$\begin{aligned}|A\cup G| &\geq \Big|A\cup\Big(\bigcup_{n=1}^{m} I_n\Big)\Big|\\ &= |A| + \sum_{n=1}^{m}\ell(I_n).\end{aligned}$$

Thus

$$\begin{aligned}|A\cup G| &\geq |A| + \sum_{n=1}^{\infty}\ell(I_n)\\ &\geq |A|+|G|,\end{aligned}$$

completing the proof that $|A\cup G| = |A|+|G|$. ■

The next result shows that the outer measure of the disjoint union of two sets is what we expect if at least one of the two sets is closed.

2.63 ***additivity of outer measure if one of the sets is closed***

Suppose A and F are disjoint subsets of $\mathbf{R}$ and F is closed. Then

$$|A\cup F| = |A| + |F|.$$

Proof Suppose $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains $A\cup F$. Let $G = \bigcup_{k=1}^{\infty} I_k$. Thus G is an open set with $A\cup F\subset G$. Hence $A\subset G\setminus F$, which implies that

2.64 $$|A| \leq |G\setminus F|.$$

Because $G\setminus F = G\cap(\mathbf{R}\setminus F)$, we know that $G\setminus F$ is an open set. Hence we can apply 2.62 to the disjoint union $G = F\cup(G\setminus F)$, getting

$$|G| = |F| + |G\setminus F|.$$

Adding $|F|$ to both sides of 2.64 and then using the equation above gives

$$\begin{aligned}|A|+|F| &\leq |G|\\ &\leq \sum_{k=1}^{\infty}\ell(I_k).\end{aligned}$$

Thus $|A|+|F| \leq |A\cup F|$, which implies that $|A|+|F| = |A\cup F|$. ■

Recall that the collection of Borel sets is the smallest σ-algebra on $\mathbf{R}$ that contains all open subsets of $\mathbf{R}$. The next result provides an extremely useful tool for approximating a Borel set by a closed set.

2.65 ***approximation of Borel sets from below by closed sets***

Suppose $B\subset\mathbf{R}$ is a Borel set. Then for every $\varepsilon > 0$, there exists a closed set $F\subset B$ such that $|B\setminus F| < \varepsilon$.

Proof Let

$$\mathcal{L} = \{D \subset \mathbf{R} : \text{for every } \varepsilon > 0, \text{ there exists a closed set } F \subset D \text{ such that } |D \setminus F| < \varepsilon\}.$$

The strategy of the proof is to show that $\mathcal{L}$ is a σ-algebra. Then because $\mathcal{L}$ contains every closed subset of $\mathbf{R}$ (if $D \subset \mathbf{R}$ is closed, take $F = D$ in the definition of $\mathcal{L}$), by taking complements we can conclude that $\mathcal{L}$ contains every open subset of $\mathbf{R}$ and thus every Borel subset of $\mathbf{R}$.

To get started with proving that $\mathcal{L}$ is a σ-algebra, we want to prove that $\mathcal{L}$ is closed under countable intersections. Thus suppose $D_1, D_2, \ldots$ is a sequence in $\mathcal{L}$. Let $\varepsilon > 0$. For each $k \in \mathbf{Z}^+$, there exists a closed set F_k such that

$$F_k \subset D_k \quad \text{and} \quad |D_k \setminus F_k| < \frac{\varepsilon}{2^k}.$$

Thus $\bigcap_{k=1}^{\infty} F_k$ is a closed set and

$$\bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} D_k \quad \text{and} \quad \Bigl(\bigcap_{k=1}^{\infty} D_k\Bigr) \setminus \Bigl(\bigcap_{k=1}^{\infty} F_k\Bigr) \subset \bigcup_{k=1}^{\infty} (D_k \setminus F_k).$$

The last set inclusion and the countable subadditivity of outer measure (see 2.8) imply that

$$\Bigl|\Bigl(\bigcap_{k=1}^{\infty} D_k\Bigr) \setminus \Bigl(\bigcap_{k=1}^{\infty} F_k\Bigr)\Bigr| < \varepsilon.$$

Thus $\bigcap_{k=1}^{\infty} D_k \in \mathcal{L}$, proving that $\mathcal{L}$ is closed under countable intersections.

Now we want to prove that $\mathcal{L}$ is closed under complementation. Suppose $D \in \mathcal{L}$ and $\varepsilon > 0$. We want to show that there is a closed subset of $\mathbf{R} \setminus D$ whose set difference with $\mathbf{R} \setminus D$ has outer measure less than ε, which will allow us to conclude that $\mathbf{R} \setminus D \in \mathcal{L}$.

First we consider the case where $|D| < \infty$. Let $F \subset D$ be a closed set such that $|D \setminus F| < \frac{\varepsilon}{2}$. The definition of outer measure implies that there exists an open set G such that $D \subset G$ and $|G| < |D| + \frac{\varepsilon}{2}$. Now $\mathbf{R} \setminus G$ is a closed set and $\mathbf{R} \setminus G \subset \mathbf{R} \setminus D$. Also, we have

$$\begin{aligned}(\mathbf{R} \setminus D) \setminus (\mathbf{R} \setminus G) &= G \setminus D \\ &\subset G \setminus F.\end{aligned}$$

Thus

$$\begin{aligned}|(\mathbf{R} \setminus D) \setminus (\mathbf{R} \setminus G)| &\leq |G \setminus F| \\ &= |G| - |F| \\ &= (|G| - |D|) + (|D| - |F|) \\ &< \frac{\varepsilon}{2} + |D \setminus F| \\ &< \varepsilon,\end{aligned}$$

where the equality in the second line above comes from applying 2.63 to the disjoint union $G = (G \setminus F) \cup F$, and the fourth line above uses subadditivity applied to the union $D = (D \setminus F) \cup F$. The last inequality above shows that $\mathbf{R} \setminus D \in \mathcal{L}$, as desired.

Now, still assuming that $D \in \mathcal{L}$ and $\varepsilon > 0$, we consider the case where $|D| = \infty$. For $k \in \mathbf{Z}^+$, let $D_k = D \cap [-k, k]$. Because $D_k \in \mathcal{L}$ and $|D_k| < \infty$, the previous case implies that $\mathbf{R} \setminus D_k \in \mathcal{L}$. Clearly $D = \bigcup_{k=1}^{\infty} D_k$. Thus

$$\mathbf{R} \setminus D = \bigcap_{k=1}^{\infty} (\mathbf{R} \setminus D_k).$$

Because $\mathcal{L}$ is closed under countable intersections, the equation above implies that $\mathbf{R} \setminus D \in \mathcal{L}$, which completes the proof that $\mathcal{L}$ is a σ-algebra. ■

Now we can prove that the outer measure of the disjoint union of two sets is what we expect if at least one of the two sets is a Borel set.

2.66 *additivity of outer measure if one of the sets is a Borel set*

Suppose A and B are disjoint subsets of $\mathbf{R}$ and B is a Borel set. Then

$$|A \cup B| = |A| + |B|.$$

Proof Let $\varepsilon > 0$. Let F be a closed set such that $F \subset B$ and $|B \setminus F| < \varepsilon$ (see 2.65). Thus

$$\begin{aligned} |A \cup B| &\geq |A \cup F| \\ &= |A| + |F| \\ &= |A| + |B| - |B \setminus F| \\ &\geq |A| + |B| - \varepsilon, \end{aligned}$$

where the second and third lines above follow from 2.63 [use $B = (B \setminus F) \cup F$ for the third line].

Because the inequality above holds for all $\varepsilon > 0$, we have $|A \cup B| \geq |A| + |B|$, which implies that $|A \cup B| = |A| + |B|$. ■

You have probably long suspected that not every subset of $\mathbf{R}$ is a Borel set. Now we can prove this suspicion.

2.67 *existence of a subset of* R *that is not a Borel set*

There exists a set $B \subset \mathbf{R}$ such that $|B| < \infty$ and B is not a Borel set.

Proof In the proof of 2.18, we showed that there exist disjoint sets $A, B \subset \mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$. For any such sets, we must have $|B| < \infty$ because otherwise both $|A \cup B|$ and $|A| + |B|$ equal ∞ (as follows from the inequality $|B| \leq |A \cup B|$). Now 2.66 implies that B is not a Borel set. ■

The tools we have constructed now allow us to prove that outer measure, when restricted to the Borel sets, is a measure.

2.68 *outer measure is a measure on Borel sets*

Outer measure is a measure on $(\mathbf{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbf{R}$.

Proof Suppose $B_1, B_2, \ldots$ is a disjoint sequence of Borel subsets of $\mathbf{R}$. Then for each $n \in \mathbf{Z}^+$ we have

$$\begin{aligned}\left|\bigcup_{k=1}^{\infty} B_k\right| &\geq \left|\bigcup_{k=1}^{n} B_k\right| \\ &= \sum_{k=1}^{n} |B_k|,\end{aligned}$$

where the first line above follows from 2.5 and the last line follows from 2.66 (and induction on n). Taking the limit as $n \to \infty$, we have $\left|\bigcup_{k=1}^{\infty} B_k\right| \geq \sum_{k=1}^{\infty} |B_k|$. The inequality in the other directions follows from countable subadditivity of outer measure (2.8). Hence

$$\left|\bigcup_{k=1}^{\infty} B_k\right| = \sum_{k=1}^{\infty} |B_k|.$$

Thus outer measure is a measure on the σ-algebra of Borel subsets of $\mathbf{R}$. ■

The result above implies that the next definition makes sense.

2.69 Definition *Lebesgue measure*

Lebesgue measure is the measure on $(\mathbf{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbf{R}$, that assigns to each Borel set its outer measure.

In other words, the Lebesgue measure of a set is the same as its outer measure, except that the term *Lebesgue measure* should not be applied to arbitrary sets but only to Borel sets (and also to what are called Lebesgue measurable sets, as we will soon see). Unlike outer measure, Lebesgue measure is actually a measure, as shown in 2.68. Lebesgue measure is named in honor of its inventor, Henri Lebesgue.



The cathedral in Beauvais, the French city where Henri Lebesgue (1875–1941) was born. Much of what we call Lebesgue measure and Lebesgue integration was developed by Lebesgue in his 1902 PhD thesis. Émile Borel was Lebesgue's PhD thesis advisor.

Lebesgue Measurable Sets

We have accomplished the major goal of this section, which was to show that outer measure restricted to Borel sets is a measure. As we will see in this subsection, outer measure is actually a measure on a somewhat larger class of sets called the Lebesgue measurable sets.

The mathematics literature contains many different definitions of a Lebesgue measurable set. These definitions are all equivalent—the definition of a Lebesgue measurable set in one approach becomes a theorem in another approach. The approach chosen here has the advantage of emphasizing that a Lebesgue measurable set differs from a Borel set by a set with outer measure 0. The attitude here is that sets with outer measure 0 should be considered small sets that do not matter much.

2.70 Definition ***Lebesgue measurable set***

A set $A \subset \mathbf{R}$ is called *Lebesgue measurable* if there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

Every Borel set is Lebesgue measurable because if $A \subset \mathbf{R}$ is a Borel set, then we can take $B = A$ in the definition above.

The result below gives several equivalent conditions for being Lebesgue measurable. The equivalence of (a) and (d) is just our definition and thus is not discussed in the proof.

Although there exist Lebesgue measurable sets that are not Borel sets, you are unlikely to encounter one. The most important application of the result below is that if $A \subset \mathbf{R}$ is a Borel set, then A satisfies conditions (b), (c), (e), and (f). Condition (c) implies that every Borel set is almost a countable union of closed sets, and condition (f) implies that every Borel set is almost a countable intersection of open sets.

2.71 ***equivalences for being a Lebesgue measurable set***

Suppose $A \subset \mathbf{R}$. Then the following are equivalent:

(a) A is Lebesgue measurable.

(b) For each $\varepsilon > 0$, there exists a closed set $F \subset A$ with $|A \setminus F| < \varepsilon$.

(c) There exist closed sets $F_1, F_2, \ldots$ contained in A such that $\Bigl|A \setminus \bigcup_{k=1}^{\infty} F_k\Bigr| = 0$.

(d) There exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

(e) For each $\varepsilon > 0$, there exists an open set $G \supset A$ such that $|G \setminus A| < \varepsilon$.

(f) There exist open sets $G_1, G_2, \ldots$ containing A such that $\Bigl|\Bigl(\bigcap_{k=1}^{\infty} G_k\Bigr) \setminus A\Bigr| = 0$.

(g) There exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$.

Proof Let $\mathcal{L}$ denote the collection of sets $A \subset \mathbf{R}$ that satisfy (b). We have already proved that every Borel set is in $\mathcal{L}$ (see 2.65). As a key part of that proof, which we will freely use in this proof, we showed that $\mathcal{L}$ is a σ-algebra on $\mathbf{R}$ (see the proof of 2.65). In addition to containing the Borel sets, $\mathcal{L}$ contains every set with outer measure 0 [because if $|A| = 0$, we can take $F = \emptyset$ in (b)].

(b) $\Longrightarrow$ (c): Suppose (b) holds. Thus for each $n \in \mathbf{Z}^+$, there exists a closed set $F_n \subset A$ such that $|A \setminus F_n| < \frac{1}{n}$. Now

$$A \setminus \bigcup_{k=1}^{\infty} F_k \subset A \setminus F_n$$

for each $n \in \mathbf{Z}^+$. Thus $|A \setminus \bigcup_{k=1}^{\infty} F_k| \leq |A \setminus F_n| \leq \frac{1}{n}$ for each $n \in \mathbf{Z}^+$. Hence $|A \setminus \bigcup_{k=1}^{\infty} F_k| = 0$, completing the proof that (b) implies (c).

(c) $\Longrightarrow$ (d): Because every countable union of closed sets is a Borel set, we see that (c) implies (d).

(d) $\Longrightarrow$ (b): Suppose (d) holds. Thus there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$. Now

$$A = B \cup (A \setminus B).$$

We know that $B \in \mathcal{L}$ (because B is a Borel set) and $A \setminus B \in \mathcal{L}$ (because $A \setminus B$ has outer measure 0). Because $\mathcal{L}$ is a σ-algebra, the displayed equation above implies that $A \in \mathcal{L}$. In other words, (b) holds, completing the proof that (d) implies (b).

At this stage of the proof, we now know that (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d).

(b) $\Longrightarrow$ (e): Suppose (b) holds. Thus $A \in \mathcal{L}$. Let $\varepsilon > 0$. Then because $\mathbf{R} \setminus A \in \mathcal{L}$ (which holds because $\mathcal{L}$ is closed under complementation), there exists a closed set $F \subset \mathbf{R} \setminus A$ such that

$$|(\mathbf{R} \setminus A) \setminus F| < \varepsilon.$$

Now $\mathbf{R} \setminus F$ is an open set with $\mathbf{R} \setminus F \supset A$. Because $(\mathbf{R} \setminus F) \setminus A = (\mathbf{R} \setminus A) \setminus F$, the inequality above implies that $|(\mathbf{R} \setminus F) \setminus A| < \varepsilon$. Thus (e) holds, completing the proof that (b) implies (e).

(e) $\Longrightarrow$ (f): Suppose (e) holds. Thus for each $n \in \mathbf{Z}^+$, there exists an open set $G_n \supset A$ such that $|G_n \setminus A| < \frac{1}{n}$. Now

$$\Big(\bigcap_{k=1}^{\infty} G_k\Big) \setminus A \subset G_n \setminus A$$

for each $n \in \mathbf{Z}^+$. Thus $|\big(\bigcap_{k=1}^{\infty} G_k\big) \setminus A| \leq |G_n \setminus A| \leq \frac{1}{n}$ for each $n \in \mathbf{Z}^+$. Hence $|\big(\bigcap_{k=1}^{\infty} G_k\big) \setminus A| = 0$, completing the proof that (e) implies (f).

(f) $\Longrightarrow$ (g): Because every countable intersection of open sets is a Borel set, we see that (f) implies (g).

(g) $\Longrightarrow$ (b): Suppose (g) holds. Thus there exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$. Now

$$A = B \cap \big(\mathbf{R} \setminus (B \setminus A)\big).$$

We know that $B \in \mathcal{L}$ (because B is a Borel set) and $\mathbf{R} \setminus (B \setminus A) \in \mathcal{L}$ (because this set is the complement of a set with outer measure 0). Because $\mathcal{L}$ is a σ-algebra, the displayed equation above implies that $A \in \mathcal{L}$. In other words, (b) holds, completing the proof that (g) implies (b).

Our chain of implications now shows that (b) through (g) are all equivalent. ∎

In addition to the equivalences in the previous result, see Exercise 13 in this section for another condition that is equivalent to being Lebesgue measurable. Also see Exercise 6, which shows that a set with finite outer measure is Lebesgue measurable if and only if it is almost a finite disjoint union of bounded open intervals.

In practice, the most useful part of Exercise 6 is the result that every Borel set with finite measure is almost a finite disjoint union of bounded open intervals.

Now we can show that outer measure is a measure on the Lebesgue measurable sets.

2.72 *outer measure is a measure on Lebesgue measurable sets*

(a) The set $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbf{R}$ is a σ-algebra on $\mathbf{R}$.

(b) Outer measure is a measure on $(\mathbf{R}, \mathcal{L})$.

Proof Because (a) and (b) are equivalent in 2.71, the set $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbf{R}$ is the collection of sets satisfying (b) in 2.71. As noted in the first paragraph of the proof of 2.71, this set is a σ-algebra on $\mathbf{R}$, proving (a).

To prove the second bullet point, suppose $A_1, A_2, \ldots$ is a disjoint sequence of Lebesgue measurable sets. By the definition of Lebesgue measurable set (2.70), for each $k \in \mathbf{Z}^+$ there exists a Borel set $B_k \subset A_k$ such that $|A_k \setminus B_k| = 0$. Now

$$
\begin{aligned}
\Big|\bigcup_{k=1}^{\infty} A_k\Big| &\geq \Big|\bigcup_{k=1}^{\infty} B_k\Big| \\
&= \sum_{k=1}^{\infty} |B_k| \\
&= \sum_{k=1}^{\infty} |A_k|,
\end{aligned}
$$

where the second line above holds because $B_1, B_2, \ldots$ is a disjoint sequence of Borel sets and outer measure is a measure on the Borel sets (see 2.68); the last line above holds because $B_k \subset A_k$ and by subadditivity of outer measure (see 2.8) we have $|A_k| = |B_k \cup (A_k \setminus B_k)| \leq |B_k| + |A_k \setminus B_k| = |B_k|$.

The inequality above, combined with countable subadditivity of outer measure (see 2.8), implies that $\big|\bigcup_{k=1}^{\infty} A_k\big| = \sum_{k=1}^{\infty} |A_k|$, completing the proof of (b). ■

If A is a set with outer measure 0, then A is Lebesgue measurable (because we can take $B = \emptyset$ in the definition 2.70). Our definition of the Lebesgue measurable sets thus implies that the set of Lebesgue measurable sets is the smallest σ-algebra on $\mathbf{R}$ containing the Borel sets and the sets with outer measure 0. Thus the set of Lebesgue measurable sets is also the smallest σ-algebra on $\mathbf{R}$ containing the open sets and the sets with outer measure 0.

Because outer measure is not even finitely additive (see 2.18), 2.72(b) implies that there exist subsets of $\mathbf{R}$ that are not Lebesgue measurable.

We previously defined Lebesgue measure as outer measure restricted to the Borel sets (see 2.69). The term *Lebesgue measure* is sometimes used in mathematical literature with the meaning as we previously defined it and is sometimes used with the following meaning.

2.73 Definition ***Lebesgue measure***

Lebesgue measure is the measure on $(\mathbf{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$, that assigns to each Lebesgue measurable set its outer measure.

The two definitions of *Lebesgue measure* disagree only on the domain of the measure—is the σ-algebra the Borel sets or the Lebesgue measurable sets? You may be able to tell which is intended from the context. In this book, the domain is specified unless it is irrelevant.

If you are reading a mathematics paper and the domain for Lebesgue measure is not specified, then it probably does not matter whether you use the Borel sets or the Lebesgue measurable sets (because every Lebesgue measurable set differs from a Borel set by a set with outer measure 0, and when dealing with measures, what happens on a set with measure 0 usually does not matter). Because all sets that arise from the usual operations of analysis are Borel sets, you may want to assume that *Lebesgue measure* means outer measure on the Borel sets, unless what you are reading explicitly states otherwise.

A mathematics paper may also refer to a *measurable* subset of $\mathbf{R}$, without further explanation. Unless some other σ-algebra is clear from the context, the author probably means the Borel sets or the Lebesgue measurable sets. Again, the choice probably does not matter, but using the Borel sets can be cleaner and simpler.

The emphasis in some textbooks on Lebesgue measurable sets instead of Borel sets probably stems from the historical development of the subject, rather than from any common use of Lebesgue measurable sets that are not Borel sets.

Lebesgue measure on the Lebesgue measurable sets does have one small advantage over Lebesgue measure on the Borel sets: every subset of a set with (outer) measure 0 is Lebesgue measurable but is not necessarily a Borel set. However, any natural process that produces a subset of $\mathbf{R}$ will produce a Borel set. Thus this small advantage does not often come up in practice.

Cantor Set and Cantor Function

Every countable set has outer measure 0 (see 2.4). A reasonable question arises about whether the converse holds. In other words, is every set with outer measure 0 countable? The Cantor set, which is introduced in this subsection, provides the answer to this question.

The Cantor set also gives counterexamples to other reasonable conjectures. For example, Exercise 17 in this section shows that the sum of two sets with Lebesgue measure 0 can have positive Lebesgue measure.

2.74 Definition ***Cantor set***

The *Cantor set* C is $[0,1] \setminus (\bigcup_{n=1}^{\infty} G_n)$, where $G_1 = (\frac{1}{3}, \frac{2}{3})$ and G_n for $n > 1$ is the union of the middle-third open intervals in the intervals of $[0,1] \setminus (\bigcup_{j=1}^{n-1} G_j)$.

One way to envision the Cantor set C is to start with the interval $[0,1]$ and then consider the process that removes at each step the middle-third open intervals of all intervals left from the previous step. At the first step, we remove $G_1 = (\frac{1}{3}, \frac{2}{3})$.



G_1 is shown in red.

After that first step, we have $[0,1] \setminus G_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Thus we take the middle-third open intervals of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. In other words, we have

$$G_2 = (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}).$$



$G_1 \cup G_2$ is shown in red.

Now $[0,1] \setminus (G_1 \cup G_2) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Thus

$$G_3 = (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup (\tfrac{19}{27}, \tfrac{20}{27}) \cup (\tfrac{25}{27}, \tfrac{26}{27}).$$



$G_1 \cup G_2 \cup G_3$ is shown in red.

Base 3 representations provide a useful way to think about the Cantor set. Just as $\frac{1}{10} = 0.1 = 0.09999\ldots$ in the decimal representation, base 3 representations are not unique for fractions whose denominator is a power of 3. For example, $\frac{1}{3} = 0.1_3 = 0.02222\ldots_3$, where the subscript 3 denotes a base 3 representations.

Notice that G_1 is the set of numbers in $[0,1]$ whose base 3 representations have 1 in the first digit after the decimal point (for those numbers that have two base 3 representations, this means both such representations must have 1 in the first digit). Also, $G_1 \cup G_2$ is the set of numbers in $[0,1]$ whose base 3 representations have 1 in the first digit or the second digit after the decimal point. And so on. Hence $\bigcup_{n=1}^{\infty} G_n$ is the set of numbers in $[0,1]$ whose base 3 representations have a 1 somewhere.

Thus we have the following description of the Cantor set. In the following result, the phrase *a base* 3 *representation* indicates that if a number has two base 3 representations, then it is in the Cantor set if and only if at least one of them contains no 1s. For example, both $\frac{1}{3}$ (which equals $0.02222\ldots_3$ and equals 0.1_3) and $\frac{2}{3}$ (which equals 0.2_3 and equals $0.12222\ldots_3$) are in the Cantor set.

2.75 *base* 3 *description of the Cantor set*

The Cantor set C is the set of numbers in $[0,1]$ that have a base 3 representation containing only 0s and 2s.

The two endpoints of each interval in each G_n are in the Cantor set. However, many elements of the Cantor set are not endpoints of any interval in any G_n. For example, Exercise 14 asks you to show that $\frac{1}{4}$ and $\frac{9}{13}$ are in the Cantor set; neither of those numbers is an endpoint of any interval in any G_n. An example of an irrational number in the Cantor set is $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n!}}$.

It is unknown whether or not every number in the Cantor set is either rational or transcendental (meaning not the root of a polynomial with integer coefficients).

The next result gives some elementary properties of the Cantor set.

2.76 C *is closed, has measure* 0, *and contains no nontrivial intervals*

(a) The Cantor set is a closed subset of $\mathbf{R}$.

(b) The Cantor set has Lebesgue measure 0.

(c) The Cantor set contains no interval with more than one element.

Proof Each set G_n used in the definition of the Cantor set is a union of open intervals. Thus each G_n is open. Thus $\bigcup_{n=1}^{\infty} G_n$ is open, and hence its complement is closed. The Cantor set equals $[0,1] \cap \bigl(\mathbf{R} \setminus \bigcup_{n=1}^{\infty} G_n\bigr)$, which is the intersection of two closed sets. Thus the Cantor set is closed, completing the proof of (a).

By induction on n, each G_n is the union of 2^{n-1} disjoint open intervals, each of which has length $\frac{1}{3^n}$. Thus $|G_n| = \frac{2^{n-1}}{3^n}$. The sets $G_1, G_2, \ldots$ are disjoint. Hence

$$\begin{aligned}
\Bigl|\bigcup_{n=1}^{\infty} G_n\Bigr| &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots \\
&= \frac{1}{3}\Bigl(1 + \frac{2}{3} + \frac{4}{9} + \cdots\Bigr) \\
&= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} \\
&= 1.
\end{aligned}$$

Thus the Cantor set, which equals $[0,1] \setminus \bigcup_{n=1}^{\infty} G_n$, has Lebesgue measure $1 - 1$ [by 2.57(b)]. In other words, the Cantor set has Lebesgue measure 0, completing the proof of (b).

A set with Lebesgue measure 0 cannot contain an interval that has more than one element. Thus (b) implies (c). ■

Now we can define an amazing function.

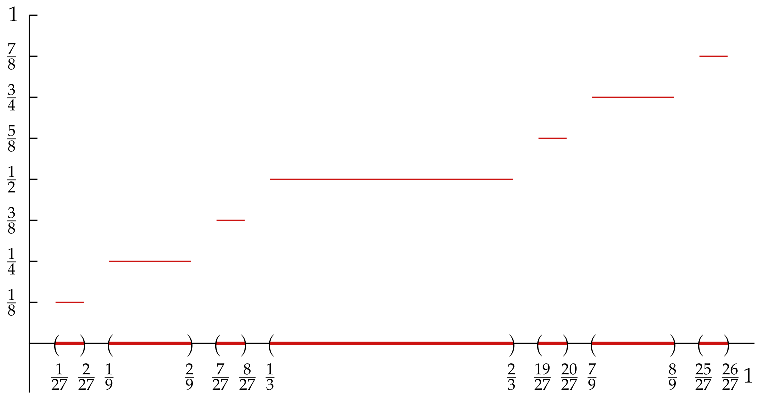
2.77 Definition ***Cantor function***

The Cantor function $\Lambda\colon [0,1] \to [0,1]$ is defined by converting base 3 representations into base 2 representations as follows:

- If $x \in C$, then $\Lambda(x)$ is computed from the unique base 3 representation of x containing only 0s and 2s by replacing each 2 by 1 and interpreting the resulting string as a base 2 number.
- If $x \in [0,1] \setminus C$, then $\Lambda(x)$ is computed from a base 3 representation of x by truncating after the first 1, replacing each 2 before the first 1 by 1, and interpreting the resulting string as a base 2 number.

2.78 Example *values of the Cantor function*

- $\Lambda(0.0202_3) = 0.0101_2$; in other words, $\Lambda\big(\frac{20}{81}\big) = \frac{5}{16}$.
- $\Lambda(0.220121_3) = 0.1101_2$; in other words $\Lambda\big(\frac{664}{729}\big) = \frac{13}{16}$.
- Suppose $x \in \big(\frac{1}{3}, \frac{2}{3}\big)$. Then $x \notin C$ because x was removed in the first step of the definition of the Cantor set. Each base 3 representation of x begins with 0.1. Thus we truncate and interpret 0.1 as a base 2 number, getting $\frac{1}{2}$. Hence the Cantor function Λ has the constant value $\frac{1}{2}$ on the interval $\big(\frac{1}{3}, \frac{2}{3}\big)$, as shown on the graph below.
- Suppose $x \in \big(\frac{7}{9}, \frac{8}{9}\big)$. Then $x \notin C$ because x was removed in the second step of the definition of the Cantor set. Each base 3 representation of x begins with 0.21. Thus we truncate, replace the 2 by 1, and interpret 0.11 as a base 2 number, getting $\frac{3}{4}$. Hence the Cantor function Λ has the constant value $\frac{3}{4}$ on the interval $\big(\frac{7}{9}, \frac{8}{9}\big)$, as shown on the graph below.



Graph of the Cantor function on the intervals from first three steps.

As shown in the next result, in some mysterious fashion the Cantor function manages to map $[0,1]$ ***onto*** $[0,1]$ even though the Cantor function is constant on each open interval in the complement of the Cantor set—see the graph in Example 2.78.

2.79 *Cantor function*

The Cantor function Λ is a continuous, increasing function from $[0,1]$ onto $[0,1]$. Furthermore, $\Lambda(C) = [0,1]$.

Proof We begin by showing that $\Lambda(C) = [0,1]$. To do this, suppose $y \in [0,1]$. In the base 2 representation of y, replace each 1 by 2 and interpret the resulting string in base 3, getting a number $x \in [0,1]$. Because x has a base 3 representation consisting only of 0s and 2s, the number x is in the Cantor set C. The definition of the Cantor function shows that $\Lambda(x) = y$. Thus $y \in \Lambda(C)$. Hence $\Lambda(C) = [0,1]$, as desired.

Some careful thinking about the meaning of base 3 and base 2 representations and the definition of the Cantor function shows that Λ is an increasing function. This step is left to the reader.

If $x \in [0,1] \setminus C$, then the Cantor function Λ is constant on an open interval containing x and thus Λ is continuous at x. If $x \in C$, then again some careful thinking about base 3 and base 2 representations shows that Λ is continuous at x.

Alternatively, you can skip the paragraph above and note that an increasing function on $[0,1]$ whose range equals $[0,1]$ is automatically continuous (although you should think about why that holds). ■

Now we can use the Cantor function to show that the Cantor set is uncountable even though it is a closed set with outer measure 0.

2.80 *C is uncountable*

The Cantor set is uncountable.

Proof If C were countable, then $\Lambda(C)$ would be countable. However, 2.79 shows that $\Lambda(C)$ is uncountable. ■

As we see in the next result, the Cantor function shows that even a continuous function can map a set with Lebesgue measure 0 to nonmeasurable sets.

2.81 *continuous image of a Lebesgue measurable set can be nonmeasurable*

There exists a Lebesgue measurable set $A \subset [0,1]$ such that $|A| = 0$ and $\Lambda(A)$ is not a Lebesgue measurable set.

Proof Let E be a subset of $[0,1]$ that is not Lebesgue measurable (the existence of such a set follows from the discussion after 2.72). Let $A = C \cap \Lambda^{-1}(E)$. Then $|A| = 0$ because $A \subset C$ and $|C| = 0$ (by 2.76). Thus A is Lebesgue measurable because every subset of $\mathbf{R}$ with Lebesgue measure 0 is Lebesgue measurable.

Because Λ maps C onto $[0,1]$ (see 2.79), we have $\Lambda(A) = E$. ■

EXERCISES 2D

1 (a) Show that the set consisting of those numbers in $(0, 1)$ that have a decimal expansion containing one hundred consecutive 4s is a Borel subset of $\mathbf{R}$.

(b) What is the Lebesgue measure of the set in part (a)?

2 Prove that there exists a bounded set $A \subset \mathbf{R}$ such that $|F| \leq |A| - 1$ for every closed set $F \subset A$.

3 Prove that there exists a set $A \subset \mathbf{R}$ such that $|G \setminus A| = \infty$ for every open set G that contains A.

4 The phrase *nontrivial interval* is used to denote an interval of $\mathbf{R}$ that contains more than one element. Recall that an interval might be open, closed, or neither.

(a) Prove that the union of each collection of nontrivial intervals of $\mathbf{R}$ is the union of a countable subset of that collection.

(b) Prove that the union of each collection of nontrivial intervals of $\mathbf{R}$ is a Borel set.

(c) Prove that there exists a collection of closed intervals of $\mathbf{R}$ whose union is not a Borel set.

5 Prove that if $A \subset \mathbf{R}$ is Lebesgue measurable, then there exists an increasing sequence $F_1 \subset F_2 \subset \cdots$ of closed sets contained in A such that

$$\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0.$$

6 Suppose $A \subset \mathbf{R}$ and $|A| < \infty$. Prove that A is Lebesgue measurable if and only if for every $\varepsilon > 0$ there exists a set G that is the union of finitely many disjoint bounded open intervals such that $|A \setminus G| + |G \setminus A| < \varepsilon$.

7 Prove that if $A \subset \mathbf{R}$ is Lebesgue measurable, then there exists a decreasing sequence $G_1 \supset G_2 \supset \cdots$ of open sets containing A such that

$$\left| \left(\bigcap_{k=1}^{\infty} G_k \right) \setminus A \right| = 0.$$

8 Prove that the collection of Lebesgue measurable subsets of $\mathbf{R}$ is translation invariant. More precisely, prove that if $A \subset \mathbf{R}$ is Lebesgue measurable and $t \in \mathbf{R}$, then $t + A$ is Lebesgue measurable.

9 Prove that the collection of Lebesgue measurable subsets of $\mathbf{R}$ is dilation invariant. More precisely, prove that if $A \subset \mathbf{R}$ is Lebesgue measurable and $t \in \mathbf{R}$, then tA (which is defined to be $\{ta : a \in A\}$) is Lebesgue measurable.

10 Prove that if A and B are disjoint subsets of $\mathbf{R}$ and B is Lebesgue measurable, then $|A \cup B| = |A| + |B|$.

11 Prove that if $A \subset \mathbf{R}$ and $|A| > 0$, then there exists a subset of A that is not Lebesgue measurable.

12 Suppose $b < c$ and $A \subset (b, c)$. Prove that A is Lebesgue measurable if and only if $|A| + |(b, c) \setminus A| = c - b$.

13 Suppose $A \subset \mathbf{R}$. Prove that A is Lebesgue measurable if and only if

$$|(-n, n) \cap A| + |(-n, n) \setminus A| = 2n$$

for every $n \in \mathbf{Z}^+$.

14 Show that $\frac{1}{4}$ and $\frac{9}{13}$ are both in the Cantor set.

15 Show that $\frac{13}{17}$ is not in the Cantor set.

16 List the eight open intervals whose union is G_4 in the definition of the Cantor set (2.74).

17 Let C denote the Cantor set. Prove that $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$.

18 Prove that every open interval of $\mathbf{R}$ contains either infinitely many or no elements in the Cantor set.

19 Evaluate $\displaystyle\int_0^1 \Lambda$, where Λ is the Cantor function.

20 Evaluate each of the following:

(a) $\Lambda\left(\frac{9}{13}\right)$;

(b) $\Lambda(0.93)$.

21 Find each of the following sets:

(a) $\Lambda^{-1}\left(\{\frac{1}{3}\}\right)$;

(b) $\Lambda^{-1}\left(\{\frac{5}{16}\}\right)$.

22 (a) Suppose x is a rational number in $[0, 1]$. Explain why $\Lambda(x)$ is rational.

(b) Suppose $x \in C$ is such that $\Lambda(x)$ is rational. Explain why x is rational.

23 Show that there exists a function $f \colon \mathbf{R} \to \mathbf{R}$ such that the image under f of every nonempty open interval is $\mathbf{R}$.

24 For $A \subset \mathbf{R}$, the quantity

$$\sup\{|F| : F \text{ is a closed bounded subset of } \mathbf{R} \text{ and } F \subset A\}$$

is called the *inner measure* of A.

(a) Show that if A is a Lebesgue measurable subset of $\mathbf{R}$, then the inner measure of A equals the outer measure of A.

(b) Show that inner measure is not a measure on the σ-algebra of all subsets of $\mathbf{R}$.

2E Convergence of Measurable Functions

Recall that a measurable space is a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X. We defined a function $f\colon X \to \mathbf{R}$ to be $\mathcal{S}$-measurable if $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \subset \mathbf{R}$. In Section 2B we proved some results about $\mathcal{S}$-measurable functions; this was before we had introduced the notion of a measure.

In this section, we return to study measurable functions, but now with an emphasis on results that depend upon measures. The highlights of this section are the proofs of Egorov's Theorem and Luzin's Theorem.

Pointwise and Uniform Convergence

We begin this section with some definitions that you probably saw in an earlier course.

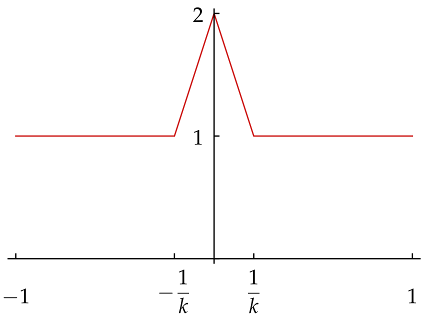
2.82 Definition ***pointwise convergence; uniform convergence***

Suppose X is a set, $f_1, f_2, \ldots$ is a sequence of functions from X to $\mathbf{R}$, and f is a function from X to $\mathbf{R}$.

- The sequence $f_1, f_2, \ldots$ *converges pointwise* on X to f if
$$\lim_{k\to\infty} f_k(x) = f(x)$$
for each $x \in X$.
In other words, $f_1, f_2, \ldots$ *converges pointwise* on X to f if for each $x \in X$ and every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $|f_k(x) - f(x)| < \varepsilon$ for all integers $k \geq n$.
- The sequence $f_1, f_2, \ldots$ *converges uniformly* on X to f if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $|f_k(x) - f(x)| < \varepsilon$ for all integers $k \geq n$ and all $x \in X$.

2.83 Example *a sequence converging pointwise but not uniformly*

Suppose $f_k\colon [-1, 1] \to \mathbf{R}$ is the function whose graph is shown here and $f\colon [-1, 1] \to \mathbf{R}$ is the function defined by
$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$
Then $f_1, f_2, \ldots$ converges pointwise on $[-1, 1]$ to f but $f_1, f_2, \ldots$ does not converge uniformly on $[-1, 1]$ to f, as you should verify.



The graph of f_k.

Like the difference between continuity and uniform continuity, the difference between pointwise convergence and uniform convergence lies in the order of the quantifiers. Take a moment to examine the definitions carefully. If a sequence of functions converges uniformly on some set, then it also converges pointwise on the same set; however, the converse is not true, as shown by Example 2.83.

Example 2.83 also shows that the pointwise limit of continuous functions need not be continuous. However, the next result tells us that the uniform limit of continuous functions is continuous.

2.84 *uniform limit of continuous functions is continuous*

Suppose $B \subset \mathbf{R}$ and $f_1, f_2, \ldots$ is a sequence of functions from B to $\mathbf{R}$ that converges uniformly on B to a function $f: B \to \mathbf{R}$. Suppose $b \in B$ and f_k is continuous at b for each $k \in \mathbf{Z}^+$. Then f is continuous at b.

Proof Suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in B$. Because f_n is continuous at b, there exists $\delta > 0$ such that $|f_n(x) - f_n(b)| < \frac{\varepsilon}{3}$ for all $x \in (b - \delta, b + \delta) \cap B$.

Now suppose $x \in (b - \delta, b + \delta) \cap B$. Then

$$\begin{aligned} |f(x) - f(b)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(b)| + |f_n(b) - f(b)| \\ &< \varepsilon. \end{aligned}$$

Thus f is continuous at b. ■

Egorov's Theorem

Dmitri Egorov (1869–1931) proved the theorem below in 1911. You may encounter some books that spell his last name as Egoroff.

A sequence of functions that converges pointwise need not converge uniformly. However, the next result says that a pointwise convergent sequence of functions on a measure space with finite total measure almost converges uniformly, in the sense that it converges uniformly except on a set that can have arbitrarily small measure.

As an example of the next result, consider Lebesgue measure λ on the interval $[-1, 1]$ and the sequence of functions $f_1, f_2, \ldots$ in Example 2.83 that converges pointwise but not uniformly on $[-1, 1]$. Suppose $\varepsilon > 0$. Then taking $E = [-1, -\frac{\varepsilon}{4}] \cup [\frac{\varepsilon}{4}, 1]$, we have $\lambda([-1, 1] \setminus E) < \varepsilon$ and $f_1, f_2, \ldots$ converges uniformly on E, as in the conclusion of the next result.

2.85 *Egorov's Theorem*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ that converges pointwise on X to a function $f: X \to \mathbf{R}$. Then for every $\varepsilon > 0$, there exists a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \varepsilon$ and $f_1, f_2, \ldots$ converges uniformly to f on E.

Proof Suppose $\varepsilon > 0$. Temporarily fix $n \in \mathbf{Z}^+$. The definition of pointwise convergence implies that

2.86
$$\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x \in X : |f_k(x) - f(x)| < \tfrac{1}{n}\} = X.$$

For $m \in \mathbf{Z}^+$, let

$$A_{m,n} = \bigcap_{k=m}^{\infty} \{x \in X : |f_k(x) - f(x)| < \tfrac{1}{n}\}.$$

Then clearly $A_{1,n} \subset A_{2,n} \subset \cdots$ is an increasing sequence of sets and 2.86 can be rewritten as

$$\bigcup_{m=1}^{\infty} A_{m,n} = X.$$

The equation above implies (by 2.59) that $\lim_{m\to\infty} \mu(A_{m,n}) = \mu(X)$. Thus there exists $m_n \in \mathbf{Z}^+$ such that

2.87
$$\mu(X) - \mu(A_{m_n,n}) < \frac{\varepsilon}{2^n}.$$

Now let

$$E = \bigcap_{n=1}^{\infty} A_{m_n,n}.$$

Then

$$\begin{aligned}\mu(X \setminus E) &= \mu\Big(X \setminus \bigcap_{n=1}^{\infty} A_{m_n,n}\Big)\\ &= \mu\Big(\bigcup_{n=1}^{\infty} (X \setminus A_{m_n,n})\Big)\\ &\leq \sum_{n=1}^{\infty} \mu(X \setminus A_{m_n,n})\\ &< \varepsilon,\end{aligned}$$

where the last inequality follows from 2.87.

To complete the proof, we must verify that $f_1, f_2, \ldots$ converges uniformly to f on E. To do this, suppose $\varepsilon' > 0$. Let $n \in \mathbf{Z}^+$ be such that $\frac{1}{n} < \varepsilon'$. Then $E \subset A_{m_n,n}$, which implies that

$$|f_k(x) - f(x)| < \tfrac{1}{n} < \varepsilon'$$

for all $k \geq m_n$ and all $x \in E$. Hence $f_1, f_2, \ldots$ does indeed converge uniformly to f on E. ∎

Approximation by Simple Functions

2.88 Definition ***simple function***

A function is called *simple* if it takes on only finitely many values.

Suppose $(X, \mathcal{S})$ is a measurable space, $f \colon X \to \mathbf{R}$ is a simple function, and $c_1, \dots, c_n$ are the distinct nonzero values of f. Then

$$f = c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n},$$

where $E_k = f^{-1}(\{c_k\})$. Thus this function f is an $\mathcal{S}$-measurable function if and only if $E_1, \dots, E_n \in \mathcal{S}$ (as you should verify).

2.89 ***approximation by simple functions***

Suppose $(X, \mathcal{S})$ is a measure space and $f \colon X \to [-\infty, \infty]$ is $\mathcal{S}$-measurable. Then there exists a sequence $f_1, f_2, \dots$ of functions from X to $\mathbf{R}$ such that

(a) each f_k is a simple $\mathcal{S}$-measurable function;

(b) $|f_k(x)| \le |f_{k+1}(x)| \le |f(x)|$ for all $k \in \mathbf{Z}^+$ and all $x \in X$;

(c) $\lim_{k \to \infty} f_k(x) = f(x)$ for every $x \in X$;

(d) $f_1, f_2, \dots$ converges uniformly on X to f if f is bounded.

Proof The idea of the proof is that for each $k \in \mathbf{Z}^+$ and $n \in \mathbf{Z}$, the interval $[n, n+1)$ is divided into 2^k equally sized half-open subintervals. If $f(x) \in [0, k)$, we define $f_k(x)$ to be the left endpoint of the subinterval into which $f(x)$ falls; if $f(x) \in (-k, 0)$, we define $f_k(x)$ to be the right endpoint of the subinterval into which $f(x)$ falls; and if $|f(x)| \ge k$, we define $f_k(x)$ to be $\pm k$. Specifically, let

$$f_k(x) = \begin{cases} \frac{m}{2^k} & \text{if } 0 \le f(x) < k \text{ and } m \in \mathbf{Z} \text{ is such that } f(x) \in \bigl[\frac{m}{2^k}, \frac{m+1}{2^k}\bigr), \\ \frac{m+1}{2^k} & \text{if } -k < f(x) < 0 \text{ and } m \in \mathbf{Z} \text{ is such that } f(x) \in \bigl[\frac{m}{2^k}, \frac{m+1}{2^k}\bigr), \\ k & \text{if } f(x) \ge k, \\ -k & \text{if } f(x) \le -k. \end{cases}$$

Each $f^{-1}\bigl(\bigl[\frac{m}{2^k}, \frac{m+1}{2^k}\bigr)\bigr) \in \mathcal{S}$ because f is an $\mathcal{S}$-measurable function. Thus each f_k is an $\mathcal{S}$-measurable simple function; in other words, (a) holds.

Also, (b) holds because of how we have defined f_k.

The definition of f_k implies that

2.90 $$|f_k(x) - f(x)| \le \frac{1}{2^k} \quad \text{for all } x \in X \text{ such that } f(x) \in [-k, k].$$

Thus we see that (c) holds.

Finally, 2.90 shows that (d) holds. ■

Luzin's Theorem

Our next result is surprising. It says that an arbitrary Borel measurable function is almost continuous, in the sense that its restriction to a large closed set is continuous. Here, the phrase *large closed set* means that we can take the complement of the closed set to have arbitrarily small measure.

Nikolai Luzin (1883–1950) proved the theorem below in 1912. Most mathematics literature in English refers to the result below as Lusin's Theorem. *However,* Luzin *is the correct transliteration from Russian into English;* Lusin *is the transliteration into German.*

Be careful about the interpretation of the conclusion of Luzin's Theorem that $f|_B$ is a continuous function on B. This is not the same as saying that f (on its original domain) is continuous at each point of B. For example, $\chi_{\mathbf{Q}}$ is discontinuous at every point of $\mathbf{R}$. However, $\chi_{\mathbf{Q}}|_{\mathbf{R}\setminus\mathbf{Q}}$ is a continuous function on $\mathbf{R}\setminus\mathbf{Q}$ (because this function is identically 0 on its domain).

2.91 *Luzin's Theorem*

Suppose $g\colon \mathbf{R}\to\mathbf{R}$ is a Borel measurable function. Then for every $\varepsilon>0$, there exists a closed set $F\subset\mathbf{R}$ such that $|\mathbf{R}\setminus F|<\varepsilon$ and $g|_F$ is a continuous function on F.

Proof First consider the special case where $g=d_1\chi_{D_1}+\cdots+d_n\chi_{D_n}$ for some distinct nonzero $d_1,\dots,d_n\in\mathbf{R}$ and some disjoint Borel sets $D_1,\dots,D_n\subset\mathbf{R}$. Suppose $\varepsilon>0$. For each $k\in\{1,\dots,n\}$, there exist (by 2.71) a closed set $F_k\subset D_k$ and an open set $G_k\supset D_k$ such that

$$|G_k\setminus D_k|<\frac{\varepsilon}{2n}\quad\text{and}\quad |D_k\setminus F_k|<\frac{\varepsilon}{2n}.$$

Because $G_k\setminus F_k=(G_k\setminus D_k)\cup(D_k\setminus F_k)$, we have $|G_k\setminus F_k|<\frac{\varepsilon}{n}$ for each $k\in\{1,\dots,n\}$.

Let

$$F=\Bigl(\bigcup_{k=1}^n F_k\Bigr)\cup\bigcap_{k=1}^n(\mathbf{R}\setminus G_k).$$

Then F is a closed subset of $\mathbf{R}$ and $\mathbf{R}\setminus F=\bigcup_{k=1}^n(G_k\setminus F_k)$. Thus $|\mathbf{R}\setminus F|<\varepsilon$.

Because $F_k\subset D_k$, we see that g is identically d_k on F_k. Thus $g|_{F_k}$ is continuous for each $k\in\{1,\dots,n\}$. Because

$$\bigcap_{k=1}^n(\mathbf{R}\setminus G_k)\subset\bigcap_{k=1}^n(\mathbf{R}\setminus D_k),$$

we see that g is identically 0 on $\bigcap_{k=1}^n(\mathbf{R}\setminus G_k)$. Thus $g|_{\bigcap_{k=1}^n(\mathbf{R}\setminus G_k)}$ is continuous. Putting all this together, we conclude that $g|_F$ is continuous (use Exercise 9 in this section), completing the proof in this special case.

Now consider an arbitrary Borel measurable function $g\colon\mathbf{R}\to\mathbf{R}$. By 2.89, there exists a sequence $g_1,g_2,\dots$ of functions from $\mathbf{R}$ to $\mathbf{R}$ that converges pointwise on $\mathbf{R}$ to g, where each g_k is a simple Borel measurable function.

Suppose $\varepsilon > 0$. By the special case already proved, for each $k \in \mathbf{Z}^+$, there exists a closed set $C_k \subset \mathbf{R}$ such that $|\mathbf{R} \setminus C_k| < \frac{\varepsilon}{2^{k+1}}$ and $g_k|_{C_k}$ is continuous. Let

$$C = \bigcap_{k=1}^{\infty} C_k.$$

Thus C is a closed set and $g_k|_C$ is continuous for every $k \in \mathbf{Z}^+$. Note that

$$\mathbf{R} \setminus C = \bigcup_{k=1}^{\infty} (\mathbf{R} \setminus C_k);$$

thus $|\mathbf{R} \setminus C| < \frac{\varepsilon}{2}$.

For each $m \in \mathbf{Z}$, the sequence $g_1|_{(m,m+1)}, g_2|_{(m,m+1)}, \ldots$ converges pointwise on $(m, m+1)$ to $g|_{(m,m+1)}$. Thus by Egorov's Theorem (2.85), for each $m \in \mathbf{Z}$, there is a Borel set $E_m \subset (m, m+1)$ such that $g_1, g_2, \ldots$ converges uniformly to g on E_m and

$$|(m, m+1) \setminus E_m| < \frac{\varepsilon}{2^{|m|+3}}.$$

Thus $g_1, g_2, \ldots$ converges uniformly to g on $C \cap E_m$ for each $m \in \mathbf{Z}$. Because each $g_k|_C$ is continuous, we conclude (using 2.84) that $g|_{C \cap E_m}$ is continuous for each $m \in \mathbf{Z}$. Thus $g|_D$ is continuous, where

$$D = \bigcup_{m \in \mathbf{Z}} (C \cap E_m).$$

Because

$$\mathbf{R} \setminus D \subset \mathbf{Z} \cup \Big(\bigcup_{m \in \mathbf{Z}} \big((m, m+1) \setminus E_m\big) \Big) \cup (\mathbf{R} \setminus C),$$

we have $|\mathbf{R} \setminus D| < \varepsilon$.

There exists a closed set $F \subset D$ such that $|D \setminus F| < \varepsilon - |\mathbf{R} \setminus D|$ (by 2.65). Now

$$|\mathbf{R} \setminus F| = |(\mathbf{R} \setminus D) \cup (D \setminus F)| \leq |\mathbf{R} \setminus D| + |D \setminus F| < \varepsilon.$$

Because the restriction of a continuous function to a smaller domain is also continuous, $g|_F$ is continuous, completing the proof. ∎

We need the following result to get another version of Luzin's Theorem.

2.92 *continuous extensions of continuous functions*

- Every continuous function on a closed subset of $\mathbf{R}$ can be extended to a continuous function on all of $\mathbf{R}$.
- More precisely, if $F \subset \mathbf{R}$ is closed and $g \colon F \to \mathbf{R}$ is continuous, then there exists a continuous function $h \colon \mathbf{R} \to \mathbf{R}$ such that $h|_F = g$.

Proof Suppose $F \subset \mathbf{R}$ is closed and $g \colon F \to \mathbf{R}$ is continuous. Thus $\mathbf{R} \setminus F$ is the union of a collection of disjoint open intervals $\{I_k\}$. For each such interval of the form (a, ∞) or of the form $(-\infty, a)$, define $h(x) = g(a)$ for all x in the interval.

For each interval I_k of the form (b, c) with $b < c$ and $b, c \in \mathbf{R}$, define h on $[b, c]$ to be the linear function such that $h(b) = g(b)$ and $h(c) = g(c)$.

Define $h(x) = g(x)$ for all $x \in \mathbf{R}$ for which $h(x)$ has not been defined by the previous two paragraphs. Then $h\colon \mathbf{R} \to \mathbf{R}$ is continuous and $h|_F = g$. ■

The next result gives a slightly modified way to state Luzin's Theorem. You can think of this version as saying that the value of a Borel measurable function can be changed on a set with small Lebesgue measure to produce a continuous function.

2.93 *Luzin's Theorem, second version*

Suppose $E \subset \mathbf{R}$ and $g\colon E \to \mathbf{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $F \subset E$ and a continuous function $h\colon \mathbf{R} \to \mathbf{R}$ such that $|E \setminus F| < \varepsilon$ and $h|_F = g|_F$.

Proof Suppose $\varepsilon > 0$. Extend g to a function $\tilde{g}\colon \mathbf{R} \to \mathbf{R}$ by defining

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbf{R} \setminus E. \end{cases}$$

By the first version of Luzin's Theorem (2.91), there is a closed set $C \subset \mathbf{R}$ such that $|\mathbf{R} \setminus C| < \varepsilon$ and $\tilde{g}|_C$ is a continuous function on C. There exists a closed set $F \subset C \cap E$ such that $|(C \cap E) \setminus F| < \varepsilon - |\mathbf{R} \setminus C|$ (by 2.65). Thus

$$|E \setminus F| \leq \big|\big((C \cap E) \setminus F\big) \cup (\mathbf{R} \setminus C)\big| \leq |(C \cap E) \setminus F| + |\mathbf{R} \setminus C| < \varepsilon.$$

Now $\tilde{g}|_F$ is a continuous function on F. Also, $\tilde{g}|_F = g|_F$ (because $F \subset E$) . Use 2.92 to extend $\tilde{g}|_F$ to a continuous function $h\colon \mathbf{R} \to \mathbf{R}$. ■



The building at Moscow State University where the mathematics seminar organized by Egorov and Luzin met. Both Egorov and Luzin had been students at Moscow State University and then later became faculty members at the same institution. Luzin's PhD thesis advisor was Egorov.

Lebesgue Measurable Functions

2.94 Definition ***Lebesgue measurable function***

A function $f\colon A \to \mathbf{R}$, where $A \subset \mathbf{R}$, is called *Lebesgue measurable* if $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set $B \subset \mathbf{R}$.

If $f\colon A \to \mathbf{R}$ is a Lebesgue measurable function, then A is a Lebesgue measurable subset of $\mathbf{R}$ [because $A = f^{-1}(\mathbf{R})$]. If A is a Lebesgue measurable subset of $\mathbf{R}$, then the definition above is the standard definition of an $\mathcal{S}$-measurable function, where $\mathcal{S}$ is a σ-algebra of all Lebesgue measurable subsets of A.

The following list summarizes and reviews some crucial definitions and results:

- A Borel set is an element of the smallest σ-algebra on $\mathbf{R}$ that contains all the open subsets of $\mathbf{R}$.
- A Lebesgue measurable set is an element of the smallest σ-algebra on $\mathbf{R}$ that contains all the open subsets of $\mathbf{R}$ and all the subsets of $\mathbf{R}$ with outer measure 0.
- The terminology *Lebesgue set* would make good sense in parallel to the terminology *Borel set*. However, *Lebesgue set* has another meaning, so we need to use *Lebesgue measurable set*.
- Every Lebesgue measurable set differs from a Borel set by a set with outer measure 0. The Borel set can be taken either to be contained in the Lebesgue measurable set or to contain the Lebesgue measurable set.
- Outer measure restricted to the σ-algebra of Borel sets is called Lebesgue measure.
- Outer measure restricted to the σ-algebra of Lebesgue measurable sets is also called Lebesgue measure.
- Outer measure is not a measure on the σ-algebra of all subsets of $\mathbf{R}$.
- A function $f\colon A \to \mathbf{R}$, where $A \subset \mathbf{R}$, is called Borel measurable if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbf{R}$.
- A function $f\colon A \to \mathbf{R}$, where $A \subset \mathbf{R}$, is called Lebesgue measurable if $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set $B \subset \mathbf{R}$.

Although there exist Lebesgue measurable sets that are not Borel sets, you are unlikely to encounter one. Similarly, a Lebesgue measurable function that is not Borel measurable is unlikely to arise in anything you do. A great way to simplify the potential confusion about Lebesgue measurable functions being defined by inverse images of Borel sets is to consider only Borel measurable functions.

"Passing from Borel to Lebesgue measurable functions is the work of the devil. Don't even consider it!" –Barry Simon (winner of the American Mathematical Society Steele Prize for Lifetime Achievement), in his five-volume series A Comprehensive Course in Analysis

The next result states that if we adopt the philosophy that what happens on a set of outer measure 0 does not matter much, then we might as well restrict our attention to Borel measurable functions.

"He professes to have received no sinister measure."
– Measure for Measure,
by William Shakespeare

2.95 *every Lebesgue measurable function is almost Borel measurable*

Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function. Then there exists a Borel measurable function $g\colon \mathbf{R} \to \mathbf{R}$ such that

$$|\{x \in \mathbf{R} : g(x) \neq f(x)\}| = 0.$$

Proof There exists a sequence $f_1, f_2, \ldots$ of Lebesgue measurable simple functions from $\mathbf{R}$ to $\mathbf{R}$ converging pointwise on $\mathbf{R}$ to f (by 2.89). Suppose $k \in \mathbf{Z}^+$. Then there exist $c_1, \ldots, c_n \in \mathbf{R}$ and disjoint Lebesgue measurable sets $A_1, \ldots, A_n \subset \mathbf{R}$ such that

$$f_k = c_1 \chi_{A_1} + \cdots + c_n \chi_{A_n}.$$

For each $j \in \{1, \ldots, n\}$, there exists a Borel set $B_j \subset A_j$ such that $|A_j \setminus B_j| = 0$ [by the equivalence of (a) and (d) in 2.71]. Let

$$g_k = c_1 \chi_{B_1} + \cdots + c_n \chi_{B_n}.$$

Then g_k is a Borel measurable function and $|\{x \in \mathbf{R} : g_k(x) \neq f_k(x)\}| = 0$.

If $x \notin \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : g_k(x) \neq f_k(x)\}$, then $g_k(x) = f_k(x)$ for all $k \in \mathbf{Z}^+$ and hence $\lim_{k \to \infty} g_k(x) = f(x)$. Let

$$E = \{x \in \mathbf{R} : \lim_{k \to \infty} g_k(x) \text{ exists in } \mathbf{R}\}.$$

Then E is a Borel subset of $\mathbf{R}$ [by Exercise 14(b) in Section 2B]. Also,

$$\mathbf{R} \setminus E \subset \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : g_k(x) \neq f_k(x)\}$$

and thus $|\mathbf{R} \setminus E| = 0$. For $x \in \mathbf{R}$, let

2.96 $$g(x) = \lim_{k \to \infty} (\chi_E g_k)(x).$$

If $x \in E$, then the limit above exists by the definition of E; if $x \in \mathbf{R} \setminus E$, then the limit above exists because $(\chi_E g_k)(x) = 0$ for all $k \in \mathbf{Z}^+$.

For each $k \in \mathbf{Z}^+$, the function $\chi_E g_k$ is Borel measurable. Thus 2.96 implies that g is a Borel measurable function (by 2.48). Because

$$\{x \in \mathbf{R} : g(x) \neq f(x)\} \subset \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : g_k(x) \neq f_k(x)\},$$

we have $|\{x \in \mathbf{R} : g(x) \neq f(x)\}| = 0$, completing the proof. ∎

EXERCISES 2E

1 Suppose X is a finite set. Explain why a sequence of functions from X to $\mathbf{R}$ that converges pointwise on X also converges uniformly on X.

2 Give an example of a sequence of functions from $\mathbf{Z}^+$ to $\mathbf{R}$ that converges pointwise on $\mathbf{Z}^+$ but does not converge uniformly on $\mathbf{Z}^+$.

3 Give an example of a sequence of continuous functions $f_1, f_2, \ldots$ from $[0,1]$ to $\mathbf{R}$ that converge pointwise to a function $f\colon [0,1] \to \mathbf{R}$ that is not a bounded function.

4 Prove or give a counterexample: If $A \subset \mathbf{R}$ and $f_1, f_2, \ldots$ is a sequence of uniformly continuous functions from A to $\mathbf{R}$ that converge uniformly to a function $f\colon A \to \mathbf{R}$, then f is uniformly continuous on A.

5 Give an example to show that Egorov's Theorem can fail without the hypothesis that $\mu(X) < \infty$.

6 Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ such that $\lim_{k\to\infty} f_k(x) = \infty$ for each $x \in X$. Prove that for every $\varepsilon > 0$, there exists a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \varepsilon$ and $f_1, f_2, \ldots$ converges uniformly to ∞ on E (meaning that for every $t > 0$, there exists $n \in \mathbf{Z}^+$ such that $f_k(x) > t$ for all integers $k \geq n$ and all $x \in E$).
[*The exercise above is an Egorov-type theorem for sequences of functions that converge pointwise to* ∞.]

7 Suppose F is a closed bounded subset of $\mathbf{R}$ and $g_1, g_2, \ldots$ is an increasing sequence of continuous real-valued functions on F (thus $g_1(x) \leq g_2(x) \leq \cdots$ for all $x \in F$) such that $\sup\{g_1(x), g_2(x), \ldots\} < \infty$ for each $x \in F$. Define a real-valued function g on F by

$$g(x) = \lim_{k\to\infty} g_k(x).$$

Prove that g is continuous on F if and only if $g_1, g_2, \ldots$ converges uniformly on F to g.
[*The result above is called Dini's Theorem.*]

8 Suppose μ is the measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ defined by

$$\mu(E) = \sum_{n\in E} \frac{1}{2^n}.$$

Prove that for every $\varepsilon > 0$, there exists a set $E \subset \mathbf{Z}^+$ with $\mu(\mathbf{Z}^+ \setminus E) < \varepsilon$ such that $f_1, f_2, \ldots$ converges uniformly on E for every sequence of functions $f_1, f_2, \ldots$ from $\mathbf{Z}^+$ to $\mathbf{R}$ that converges pointwise on $\mathbf{Z}^+$.
[*This result does not follow from Egorov's Theorem because here we are asking for E to depend only on* ε. *In Egorov's Theorem, E depends on* ε *and on the sequence* $f_1, f_2, \ldots$.]

9 Suppose $F_1, \dots, F_n$ are disjoint closed subsets of $\mathbf{R}$. Prove that if

$$g\colon F_1 \cup \cdots \cup F_n \to \mathbf{R}$$

is a function such that $g|_{F_k}$ is a continuous function for each $k \in \{1, \dots, n\}$, then g is a continuous function.

10 Suppose $F \subset \mathbf{R}$ is such that every continuous function from F to $\mathbf{R}$ can be extended to a continuous function from $\mathbf{R}$ to $\mathbf{R}$. Prove that F is a closed subset of $\mathbf{R}$.

11 Prove or give a counterexample: If $F \subset \mathbf{R}$ is such that every bounded continuous function from F to $\mathbf{R}$ can be extended to a continuous function from $\mathbf{R}$ to $\mathbf{R}$, then F is a closed subset of $\mathbf{R}$.

12 Give an example of a Borel measurable function f from $\mathbf{R}$ to $\mathbf{R}$ such that there does not exist a set $B \subset \mathbf{R}$ such that $|\mathbf{R} \setminus B| = 0$ and $f|_B$ is a continuous function on B.

13 Prove or give a counterexample: If $f_t\colon \mathbf{R} \to \mathbf{R}$ is a Borel measurable function for each $t \in \mathbf{R}$ and $f\colon \mathbf{R} \to (-\infty, \infty]$ is defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\},$$

then f is a Borel measurable function.

14 Suppose $b_1, b_2, \dots$ is a sequence of real numbers. Define $f\colon \mathbf{R} \to [0, \infty]$ by

$$f(x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{1}{4^k |x - b_k|} & \text{if } x \notin \{b_1, b_2, \dots\}, \\ \infty & \text{if } x \in \{b_1, b_2, \dots\}. \end{cases}$$

Prove that $|\{x \in \mathbf{R} : f(x) < 1\}| = \infty$.
[*This exercise is a variation of a problem originally considered by Borel. If* $b_1, b_2, \dots$ *contains all the rational numbers, then it is not even obvious that* $\{x \in \mathbf{R} : f(x) < \infty\} \neq \varnothing$.]

15 Suppose B is a Borel set and $f\colon B \to \mathbf{R}$ is a Lebesgue measurable function. Show that there exists a Borel measurable function $g\colon B \to \mathbf{R}$ such that

$$|\{x \in B : g(x) \neq f(x)\}| = 0.$$

Chapter 3
Integration

To remedy deficiencies of Riemann integration that were discussed in Section 1B, in the last chapter we developed measure theory as an extension of the notion of the length of an interval. Having proved the fundamental results about measures, we are now ready to use measures to develop integration with respect to a measure.

As we will see, this new method of integration fixes many of the problems with Riemann integration. In particular, we will develop good theorems for interchanging limits and integrals.



Statue in Milan of Maria Gaetana Agnesi,
who in 1748 published one of the first calculus textbooks.
A translation of her book into English was published in 1801.
In this chapter, we develop a method of integration more powerful
than methods contemplated by the pioneers of calculus.
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S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_3

3A Integration with Respect to a Measure

Integration of Nonnegative Functions

We will first define the integral of a nonnegative function with respect to a measure. Then by writing a real-valued function as the difference of two nonnegative functions, we will define the integral of a real-valued function with respect to a measure. We begin this process with the following definition.

3.1 Definition *$\mathcal{S}$-partition*

Suppose $\mathcal{S}$ is a σ-algebra on a set X. An *$\mathcal{S}$-partition* of X is a finite collection $A_1, \dots, A_m$ of disjoint sets in $\mathcal{S}$ such that $A_1 \cup \cdots \cup A_m = X$.

We adopt the convention that $0 \cdot \infty$ and $\infty \cdot 0$ should both be interpreted to be 0.

The next definition should remind you of the definition of the lower Riemann sum (see 1.3). However, now we are working with an arbitrary measure and thus X need not be a subset of $\mathbf{R}$. More importantly, even in the case when X is a closed interval $[a, b]$ in $\mathbf{R}$ and μ is Lebesgue measure on the Borel subsets of $[a, b]$, the sets $A_1, \dots, A_m$ in the definition below do not need to be subintervals of $[a, b]$ as they do for the lower Riemann sum—they need only be Borel sets.

3.2 Definition *lower Lebesgue sum*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f\colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function, and P is an $\mathcal{S}$-partition $A_1, \dots, A_m$ of X. The *lower Lebesgue sum* $\mathcal{L}(f, P)$ is defined by

$$\mathcal{L}(f, P) = \sum_{j=1}^{m} \mu(A_j) \inf_{A_j} f.$$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. We will denote the integral of an $\mathcal{S}$-measurable function f with respect to μ by $\int f \, d\mu$. Our basic requirements for an integral are that we want $\int \chi_E \, d\mu$ to equal $\mu(E)$ for all $E \in \mathcal{S}$, and we want $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$. As we will see, the following definition satisfies both of those requirements (although this is not obvious). Think about why the following definition is reasonable in terms of the integral equaling the area under the graph of the function (in the special case of Lebesgue measure on an interval of $\mathbf{R}$).

3.3 Definition *integral of a nonnegative function*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. The *integral* of f with respect to μ, denoted $\int f \, d\mu$, is defined by

$$\int f \, d\mu = \sup\{\mathcal{L}(f, P) : P \text{ is an } \mathcal{S}\text{-partition of } X\}.$$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Each $\mathcal{S}$-partition $A_1, \dots, A_m$ of X leads to an approximation of f from below by the $\mathcal{S}$-measurable simple function $\sum_{j=1}^{m} \bigl(\inf_{A_j} f\bigr) \chi_{A_j}$. This suggests that

$$\sum_{j=1}^{m} \mu(A_j) \inf_{A_j} f$$

should be an approximation from below of our intuitive notion of $\int f \, d\mu$. Taking the supremum of these approximations leads to our definition of $\int f \, d\mu$.

The following result gives our first example of evaluating an integral.

3.4 ***integral of a characteristic function***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E \in \mathcal{S}$. Then

$$\int \chi_E \, d\mu = \mu(E).$$

Proof If P is the $\mathcal{S}$-partition of X consisting of E and its complement $X \setminus E$, then clearly $\mathcal{L}(\chi_E, P) = \mu(E)$. Thus $\int \chi_E \, d\mu \geq \mu(E)$.

To prove the inequality in the other direction, suppose P is an $\mathcal{S}$-partition $A_1, \dots, A_m$ of X. Then $\mu(A_j) \inf_{A_j} \chi_E$ equals $\mu(A_j)$ if $A_j \subset E$ and equals 0 otherwise. Thus

$$\begin{aligned} \mathcal{L}(\chi_E, P) &= \sum_{\{j \colon A_j \subset E\}} \mu(A_j) \\ &= \mu\Bigl(\bigcup_{\{j \colon A_j \subset E\}} A_j\Bigr) \\ &\leq \mu(E). \end{aligned}$$

Thus $\int \chi_E \, d\mu \leq \mu(E)$, completing the proof. ∎

The symbol d in the expression $\int f \, d\mu$ has no independent meaning, but it often usefully separates f from μ. Because the d in $\int f \, d\mu$ does not represent another object, some mathematicians prefer typesetting an upright d in this situation, producing $\int f \, \mathrm{d}\mu$. However, the upright d looks jarring to some readers who are accustomed to italicized symbols. This book takes the compromise position of using slanted d instead of math-mode italicized d in integrals.

3.5 Example *integrals of $\chi_{\mathbf{Q}}$ and $\chi_{[0,1] \setminus \mathbf{Q}}$*

Suppose λ is Lebesgue measure on $\mathbf{R}$. As a special case of the result above, we have $\int \chi_{\mathbf{Q}} \, d\lambda = 0$ (because $|\mathbf{Q}| = 0$). Recall that $\chi_{\mathbf{Q}}$ is not Riemann integrable on $[0, 1]$. Thus even at this early stage in our development of integration with respect to a measure, we have fixed one of the deficiencies of Riemann integration.

Note also that 3.4 implies that $\int \chi_{[0,1] \setminus \mathbf{Q}} \, d\lambda = 1$ (because $|[0, 1] \setminus \mathbf{Q}| = 1$), which is what we want. In contrast, the lower Riemann integral of $\chi_{[0,1] \setminus \mathbf{Q}}$ on $[0, 1]$ equals 0, which is not what we want.

3.6 Example *integration with respect to counting measure is summation*

Suppose μ is counting measure on $\mathbf{Z}^+$ and $b_1, b_2, \ldots$ is a sequence of nonnegative numbers. Think of b as the function from $\mathbf{Z}^+$ to $[0, \infty)$ defined by $b(k) = b_k$. Then

$$\int b \, d\mu = \sum_{k=1}^{\infty} b_k,$$

as you should verify.

Integration with respect to a measure can be called *Lebesgue integration*. The next result shows that Lebesgue integration behaves as expected on simple functions represented as linear combinations of characteristic functions of disjoint sets.

3.7 ***integral of a simple function***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $E_1, \ldots, E_n$ are disjoint sets in $\mathcal{S}$, and $c_1, \ldots, c_n \in [0, \infty]$. Then

$$\int \Bigl(\sum_{k=1}^{n} c_k \chi_{E_k}\Bigr) d\mu = \sum_{k=1}^{n} c_k \mu(E_k).$$

Proof Without loss of generality, we can assume that $E_1, \ldots, E_n$ is an $\mathcal{S}$-partition of X [by replacing n by $n+1$ and setting $E_{n+1} = X \setminus (E_1 \cup \ldots \cup E_n)$ and $c_{n+1} = 0$].

If P is the $\mathcal{S}$-partition $E_1, \ldots, E_n$ of X, then $\mathcal{L}\bigl(\sum_{k=1}^{n} c_k \chi_{E_k}, P\bigr) = \sum_{k=1}^{n} c_k \mu(E_k)$. Thus

$$\int \Bigl(\sum_{k=1}^{n} c_k \chi_{E_k}\Bigr) d\mu \geq \sum_{k=1}^{n} c_k \mu(E_k).$$

To prove the inequality in the other direction, suppose that P is an $\mathcal{S}$-partition $A_1, \ldots, A_m$ of X. Then

$$\begin{aligned}
\mathcal{L}\Bigl(\sum_{k=1}^{n} c_k \chi_{E_k}, P\Bigr) &= \sum_{j=1}^{m} \mu(A_j) \min_{\{i \,:\, A_j \cap E_i \neq \varnothing\}} c_i \\
&= \sum_{j=1}^{m} \sum_{k=1}^{n} \mu(A_j \cap E_k) \min_{\{i \,:\, A_j \cap E_i \neq \varnothing\}} c_i \\
&\leq \sum_{j=1}^{m} \sum_{k=1}^{n} \mu(A_j \cap E_k) c_k \\
&= \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu(A_j \cap E_k) \\
&= \sum_{k=1}^{n} c_k \mu(E_k).
\end{aligned}$$

The inequality above implies that $\int \bigl(\sum_{k=1}^{n} c_k \chi_{E_k}\bigr) d\mu \leq \sum_{k=1}^{n} c_k \mu(E_k)$, completing the proof. ∎

The next easy result gives an unsurprising property of integrals.

3.8 ***integration is order preserving***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, g \colon X \to [0, \infty]$ are $\mathcal{S}$-measurable functions such that $f(x) \leq g(x)$ for all $x \in X$. Then $\int f \, d\mu \leq \int g \, d\mu$.

Proof Suppose P is an $\mathcal{S}$-partition $A_1, \ldots, A_m$ of X. Then

$$\inf_{A_j} f \leq \inf_{A_j} g$$

for each $j = 1, \ldots, m$. Thus $\mathcal{L}(f, P) \leq \mathcal{L}(g, P)$. Hence $\int f \, d\mu \leq \int g \, d\mu$. ■

Monotone Convergence Theorem

For the proof of the Monotone Convergence Theorem (and several other results), we will need to use the following mild restatement of the definition of the integral of a nonnegative function.

3.9 ***integrals via simple functions***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \colon X \to [0, \infty]$ is $\mathcal{S}$-measurable. Then

3.10 $$\int f \, d\mu = \sup\Big\{ \sum_{j=1}^{m} c_j \mu(A_j) : A_1, \ldots, A_m \text{ are disjoint sets in } \mathcal{S}, \; c_1, \ldots, c_m \in [0, \infty), \text{ and } f(x) \geq \sum_{j=1}^{m} c_j \chi_{A_j}(x) \text{ for every } x \in X \Big\}.$$

Proof First note that the left side of 3.10 is bigger than or equal to the right side by 3.7 and 3.8.

To prove that the right side of 3.10 is bigger than or equal to the left side, first assume that $\inf_A f < \infty$ for every $A \in \mathcal{S}$ with $\mu(A) > 0$. Then for P an $\mathcal{S}$-partition $A_1, \ldots, A_m$ of nonempty subsets of X, take $c_j = \inf_{A_j} f$, which shows that $\mathcal{L}(f, P)$ is in the set on the right side of 3.10. Thus the definition of $\int f \, d\mu$ shows that the right side of 3.10 is bigger than or equal to the left side.

The only remaining case to consider is when there exists a set $A \in \mathcal{S}$ such that $\mu(A) > 0$ and $\inf_A f = \infty$ [which implies that $f(x) = \infty$ for all $x \in A$]. In this case, for arbitrary $t \in (0, \infty)$ we can take $m = 1$, $A_1 = A$, and $c_1 = t$. These choices show that the right side of 3.10 is at least $t\mu(A)$. Because t is an arbitrary positive number, this shows that the right side of 3.10 equals ∞, which of course is greater than or equal to the left side, completing the proof. ■

The next result allows us to interchange limits and integrals in certain circumstances. We will see more theorems of this nature in the next section.

3.11 *Monotone Convergence Theorem*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 \leq f_1 \leq f_2 \leq \cdots$ is an increasing sequence of $\mathcal{S}$-measurable functions. Define $f \colon X \to [0, \infty]$ by

$$f(x) = \lim_{k \to \infty} f_k(x).$$

Then

$$\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu.$$

Proof The function f is $\mathcal{S}$-measurable by 2.53.

Because $f_k(x) \leq f(x)$ for every $x \in X$, we have $\int f_k \, d\mu \leq \int f \, d\mu$ for each $k \in \mathbf{Z}^+$ (by 3.8). Thus $\lim_{k \to \infty} \int f_k \, d\mu \leq \int f \, d\mu$.

To prove the inequality in the other direction, suppose $A_1, \dots, A_m$ are disjoint sets in $\mathcal{S}$ and $c_1, \dots, c_m \in [0, \infty)$ are such that

3.12 $$f(x) \geq \sum_{j=1}^{m} c_j \chi_{A_j}(x) \quad \text{for every } x \in X.$$

Let $t \in (0, 1)$. For $k \in \mathbf{Z}^+$, let

$$E_k = \Big\{ x \in X : f_k(x) \geq t \sum_{j=1}^{m} c_j \chi_{A_j}(x) \Big\}.$$

Then $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{S}$ whose union equals X. Thus $\lim_{k \to \infty} \mu(A_j \cap E_k) = \mu(A_j)$ for each $j \in \{1, \dots, m\}$ (by 2.59).

If $k \in \mathbf{Z}^+$, then

$$f_k(x) \geq \sum_{j=1}^{m} t c_j \chi_{A_j \cap E_k}(x)$$

for every $x \in X$. Thus (by 3.9)

$$\int f_k \, d\mu \geq t \sum_{j=1}^{m} c_j \mu(A_j \cap E_k).$$

Taking the limit as $k \to \infty$ of both sides of the inequality above gives

$$\lim_{k \to \infty} \int f_k \, d\mu \geq t \sum_{j=1}^{m} c_j \mu(A_j).$$

Now taking the limit as t increases to 1 shows that

$$\lim_{k \to \infty} \int f_k \, d\mu \geq \sum_{j=1}^{m} c_j \mu(A_j).$$

Taking the supremum of the inequality above over all $\mathcal{S}$-partitions $A_1, \dots, A_m$ of X and all $c_1, \dots, c_m \in [0, \infty)$ satisfying 3.12 shows (using 3.9) that we have $\lim_{k \to \infty} \int f_k \, d\mu \geq \int f \, d\mu$, completing the proof. ■

The proof that the integral is additive will use the Monotone Convergence Theorem and our next result. The representation of a simple function $h\colon X \to [0,\infty]$ in the form $\sum_{k=1}^{n} c_k \chi_{E_k}$ is not unique. Requiring the numbers $c_1, \dots, c_n$ to be distinct and $E_1, \dots, E_n$ to be nonempty and disjoint with $E_1 \cup \dots \cup E_n = X$ produces what is called the *standard representation* of a simple function [take $E_k = h^{-1}(\{c_k\})$, where $c_1, \dots, c_n$ are the distinct values of h]. The following lemma shows that all representations (including representations with sets that are not disjoint) of a simple measurable function give the same sum that we expect from integration.

3.13 *integral-type sums for simple functions*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Suppose $a_1, \dots, a_m, b_1, \dots, b_n \in [0,\infty]$ and $A_1, \dots, A_m, B_1, \dots, B_n \in \mathcal{S}$ are such that $\sum_{j=1}^{m} a_j \chi_{A_j} = \sum_{k=1}^{n} b_k \chi_{B_k}$. Then

$$\sum_{j=1}^{m} a_j \mu(A_j) = \sum_{k=1}^{n} b_k \mu(B_k).$$

Proof We assume $A_1 \cup \dots \cup A_m = X$ (otherwise add the term $0\chi_{X \setminus (A_1 \cup \dots \cup A_m)}$).

Suppose A_1 and A_2 are not disjoint. Then we can write

3.14 $$a_1 \chi_{A_1} + a_2 \chi_{A_2} = a_1 \chi_{A_1 \setminus A_2} + a_2 \chi_{A_2 \setminus A_1} + (a_1 + a_2) \chi_{A_1 \cap A_2},$$

where the three sets appearing on the right side of the equation above are disjoint.

Now $A_1 = (A_1 \setminus A_2) \cup (A_1 \cap A_2)$ and $A_2 = (A_2 \setminus A_1) \cup (A_1 \cap A_2)$; each of these unions is a disjoint union. Thus $\mu(A_1) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2)$ and $\mu(A_2) = \mu(A_2 \setminus A_1) + \mu(A_1 \cap A_2)$. Hence

$$a_1 \mu(A_1) + a_2 \mu(A_2) = a_1 \mu(A_1 \setminus A_2) + a_2 \mu(A_2 \setminus A_1) + (a_1 + a_2) \mu(A_1 \cap A_2).$$

The equation above, in conjunction with 3.14, shows that if we replace the two sets A_1, A_2 by the three disjoint sets $A_1 \setminus A_2, A_2 \setminus A_1, A_1 \cap A_2$ and make the appropriate adjustments to the coefficients $a_1, \dots, a_m$, then the value of the sum $\sum_{j=1}^{m} a_j \mu(A_j)$ is unchanged (although m has increased by 1).

Repeating this process with all pairs of subsets among $A_1, \dots, A_m$ that are not disjoint after each step, in a finite number of steps we can convert the initial list $A_1, \dots, A_m$ into a disjoint list of subsets without changing the value of $\sum_{j=1}^{m} a_j \mu(A_j)$.

The next step is to make the numbers $a_1, \dots, a_m$ distinct. This is done by replacing the sets corresponding to each a_j by the union of those sets, and using finite additivity of the measure μ to show that the value of the sum $\sum_{j=1}^{m} a_j \mu(A_j)$ does not change.

Finally, drop any terms for which $A_j = \varnothing$, getting the standard representation for a simple function. We have now shown that the original value of $\sum_{j=1}^{m} a_j \mu(A_j)$ is equal to the value if we use the standard representation of the simple function $\sum_{j=1}^{m} a_j \chi_{A_j}$. The same procedure can be used with the representation $\sum_{k=1}^{n} b_k \chi_{B_k}$ to show that $\sum_{k=1}^{n} b_k \mu(\chi_{B_k})$ equals what we would get with the standard representation. Thus the equality of the functions $\sum_{j=1}^{m} a_j \chi_{A_j}$ and $\sum_{k=1}^{n} b_k \chi_{B_k}$ implies the equality $\sum_{j=1}^{m} a_j \mu(A_j) = \sum_{k=1}^{n} b_k \mu(B_k)$. ■

Now we can show that our definition of integration does the right thing with simple measurable functions that might not be expressed in the standard representation. The result below differs from 3.7 mainly because the sets $E_1, \dots, E_n$ in the result below are not required to be disjoint. Like the previous result, the next result would follow immediately from the linearity of integration if that property had already been proved.

If we had already proved that integration is linear, then we could quickly get the conclusion of the previous result by integrating both sides of the equation $\sum_{j=1}^m a_j \chi_{A_j} = \sum_{k=1}^n b_k \chi_{B_k}$ with respect to μ. However, we need the previous result to prove the next result, which is used in our proof that integration is linear.

3.15 *integral of a linear combination of characteristic functions*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $E_1, \dots, E_n \in \mathcal{S}$, and $c_1, \dots, c_n \in [0, \infty]$. Then

$$\int \Bigl(\sum_{k=1}^n c_k \chi_{E_k}\Bigr)\, d\mu = \sum_{k=1}^n c_k \mu(E_k).$$

Proof The desired result follows from writing the simple function $\sum_{k=1}^n c_k \chi_{E_k}$ in the standard representation for a simple function and then using 3.7 and 3.13. ■

Now we can prove that integration is additive on nonnegative functions.

3.16 *additivity of integration*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, g \colon X \to [0, \infty]$ are $\mathcal{S}$-measurable functions. Then

$$\int (f + g)\, d\mu = \int f\, d\mu + \int g\, d\mu.$$

Proof The desired result holds for simple nonnegative $\mathcal{S}$-measurable functions (by 3.15). Thus we approximate by such functions.

Specifically, let $f_1, f_2, \dots$ and $g_1, g_2, \dots$ be increasing sequences of simple nonnegative $\mathcal{S}$-measurable functions such that

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{and} \quad \lim_{k\to\infty} g_k(x) = g(x)$$

for all $x \in X$ (see 2.89 for the existence of such increasing sequences). Then

$$\begin{aligned}
\int (f+g)\, d\mu &= \lim_{k\to\infty} \int (f_k + g_k)\, d\mu \\
&= \lim_{k\to\infty} \int f_k\, d\mu + \lim_{k\to\infty} \int g_k\, d\mu \\
&= \int f\, d\mu + \int g\, d\mu,
\end{aligned}$$

where the first and third equalities follow from the Monotone Convergence Theorem and the second equality holds by 3.15. ■

The lower Riemann integral is not additive, even for bounded nonnegative measurable functions. For example, if $f = \chi_{\mathbf{Q} \cap [0,1]}$ and $g = \chi_{[0,1] \setminus \mathbf{Q}}$, then

$$L(f, [0,1]) = 0 \quad \text{and} \quad L(g, [0,1]) = 0 \quad \text{but} \quad L(f+g, [0,1]) = 1.$$

In contrast, if λ is Lebesgue measure on the Borel subsets of $[0,1]$, then

$$\int f \, d\lambda = 0 \quad \text{and} \quad \int g \, d\lambda = 1 \quad \text{and} \quad \int (f+g) \, d\lambda = 1.$$

More generally, we have just proved that $\int (f+g)\, d\mu = \int f \, d\mu + \int g \, d\mu$ for every measure μ and for all nonnegative measurable functions f and g. Recall that integration with respect to a measure is defined via lower Lebesgue sums in a similar fashion to the definition of the lower Riemann integral via lower Riemann sums (with the big exception of allowing measurable sets instead of just intervals in the partitions). However, we have just seen that the integral with respect to a measure (which could have been called the lower Lebesgue integral) has considerably nicer behavior (additivity!) than the lower Riemann integral.

Integration of Real-Valued Functions

The following definition gives us a standard way to write an arbitrary real-valued function as the difference of two nonnegative functions.

3.17 Definition f^+; f^-

Suppose $f \colon X \to [-\infty, \infty]$ is a function. Define functions f^+ and f^- from X to $[0, \infty]$ by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0 \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0, \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Note that if $f \colon X \to [-\infty, \infty]$ is a function, then

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

The decomposition above allows us to extend our definition of integration to functions that take on negative as well as positive values.

3.18 Definition ***integral of a real-valued function***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \colon X \to [-\infty, \infty]$ is an $\mathcal{S}$-measurable function such that at least one of $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ is finite. The *integral* of f with respect to μ, denoted $\int f \, d\mu$, is defined by

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

If $f \geq 0$, then $f^+ = f$ and $f^- = 0$; thus this definition is consistent with the previous definition of the integral of a nonnegative function.

The condition $\int |f|\,d\mu < \infty$ is equivalent to the condition $\int f^+\,d\mu < \infty$ and $\int f^-\,d\mu < \infty$ (because $|f| = f^+ + f^-$).

3.19 Example *a function whose integral is not defined*

Suppose λ is Lebesgue measure on $\mathbf{R}$ and $f\colon \mathbf{R} \to \mathbf{R}$ is the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then $\int f\,d\lambda$ is not defined because $\int f^+\,d\lambda = \infty$ and $\int f^-\,d\lambda = \infty$.

The next result says that the integral of a number times a function is exactly what we expect.

3.20 ***integration is homogeneous***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [-\infty, \infty]$ is a function such that $\int f\,d\mu$ is defined. If $c \in \mathbf{R}$, then

$$\int cf\,d\mu = c\int f\,d\mu.$$

Proof First consider the case where f is a nonnegative function and $c \geq 0$. If P is an $\mathcal{S}$-partition of X, then clearly $\mathcal{L}(cf, P) = c\mathcal{L}(f, P)$. Thus $\int cf\,d\mu = c\int f\,d\mu$.

Now consider the general case where f takes values in $[-\infty, \infty]$. Suppose $c \geq 0$. Then

$$\begin{aligned} \int cf\,d\mu &= \int (cf)^+\,d\mu - \int (cf)^-\,d\mu \\ &= \int cf^+\,d\mu - \int cf^-\,d\mu \\ &= c\Big(\int f^+\,d\mu - \int f^-\,d\mu\Big) \\ &= c\int f\,d\mu, \end{aligned}$$

where the third line follows from the first paragraph of this proof.

Finally, now suppose $c < 0$ (still assuming that f takes values in $[-\infty, \infty]$). Then $-c > 0$ and

$$\begin{aligned} \int cf\,d\mu &= \int (cf)^+\,d\mu - \int (cf)^-\,d\mu \\ &= \int (-c)f^-\,d\mu - \int (-c)f^+\,d\mu \\ &= (-c)\Big(\int f^-\,d\mu - \int f^+\,d\mu\Big) \\ &= c\int f\,d\mu, \end{aligned}$$

completing the proof. ∎

Now we prove that integration with respect to a measure has the additive property required for a good theory of integration.

3.21 *additivity of integration*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, g \colon X \to \mathbf{R}$ are $\mathcal{S}$-measurable functions such that $\int |f| \, d\mu < \infty$ and $\int |g| \, d\mu < \infty$. Then

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Proof Clearly

$$\begin{aligned}(f+g)^+ - (f+g)^- &= f+g \\ &= f^+ - f^- + g^+ - g^-.\end{aligned}$$

Thus

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Both sides of the equation above are sums of nonnegative functions. Thus integrating both sides with respect to μ and using 3.16 gives

$$\int (f+g)^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int (f+g)^- \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu.$$

Rearranging the equation above gives

$$\int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu,$$

where the left side is not of the form $\infty - \infty$ because $(f+g)^+ \leq f^+ + g^+$ and $(f+g)^- \leq f^- + g^-$. The equation above can be rewritten as

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu,$$

completing the proof. ■

Gottfried Leibniz (1646–1716) invented the symbol $\int$ to denote integration in 1675.

The next result resembles 3.8, but now the functions are allowed to be real valued.

3.22 *integration is order preserving*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, g \colon X \to \mathbf{R}$ are $\mathcal{S}$-measurable functions such that $\int f \, d\mu$ and $\int g \, d\mu$ are defined. Suppose also that $f(x) \leq g(x)$ for all $x \in X$. Then $\int f \, d\mu \leq \int g \, d\mu$.

Proof The cases where $\int f \, d\mu = \pm\infty$ or $\int g \, d\mu = \pm\infty$ are left to the reader. Thus we assume that $\int |f| \, d\mu < \infty$ and $\int |g| \, d\mu < \infty$.

The additivity (3.21) and homogeneity (3.20 with $c = -1$) of integration imply that

$$\int g \, d\mu - \int f \, d\mu = \int (g-f) \, d\mu.$$

The last integral is nonnegative because $g(x) - f(x) \geq 0$ for all $x \in X$. ■

The inequality in the next result receives frequent use.

3.23 *absolute value of integral $\le$ integral of absolute value*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [-\infty, \infty]$ is a function such that $\int f\, d\mu$ is defined. Then

$$\left|\int f\, d\mu\right| \le \int |f|\, d\mu.$$

Proof Because $\int f\, d\mu$ is defined, f is an $\mathcal{S}$-measurable function and at least one of $\int f^+\, d\mu$ and $\int f^-\, d\mu$ is finite. Thus

$$\begin{aligned}
\left|\int f\, d\mu\right| &= \left|\int f^+\, d\mu - \int f^-\, d\mu\right| \\
&\le \int f^+\, d\mu + \int f^-\, d\mu \\
&= \int (f^+ + f^-)\, d\mu \\
&= \int |f|\, d\mu,
\end{aligned}$$

as desired. ■

EXERCISES 3A

1 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function such that $\int f\, d\mu < \infty$. Explain why

$$\inf_E f = 0$$

for each set $E \in \mathcal{S}$ with $\mu(E) = \infty$.

2 Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $c \in X$. Define the Dirac measure δ_c on $(X, \mathcal{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

Prove that if $f\colon X \to [0, \infty]$ is $\mathcal{S}$-measurable, then $\int f\, d\delta_c = f(c)$.
[*Careful: $\{c\}$ may not be in $\mathcal{S}$.*]

3 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Prove that

$$\int f\, d\mu > 0 \text{ if and only if } \mu(\{x \in X : f(x) > 0\}) > 0.$$

4 Give an example of a Borel measurable function $f\colon [0,1] \to (0,\infty)$ such that $L(f,[0,1]) = 0$.
[*Recall that $L(f,[0,1])$ denotes the lower Riemann integral, which was defined in Section 1A. If λ is Lebesgue measure on $[0,1]$, then the previous exercise states that $\int f\,d\lambda > 0$ for this function f, which is what we expect of a positive function. Thus even though both $L(f,[0,1])$ and $\int f\,d\lambda$ are defined by taking the supremum of approximations from below, Lebesgue measure captures the right behavior for this function f and the lower Riemann integral does not.*]

5 Verify the assertion that integration with respect to counting measure is summation (Example 3.6).

6 Suppose $(X,\mathcal{S},\mu)$ is a measure space, $f\colon X \to [0,\infty]$ is $\mathcal{S}$-measurable, and P and P' are $\mathcal{S}$-partitions of X such that each set in P' is contained in some set in P. Prove that $\mathcal{L}(f,P) \le \mathcal{L}(f,P')$.

7 Suppose X is a set, $\mathcal{S}$ is the σ-algebra of all subsets of X, and $w\colon X \to [0,\infty]$ is a function. Define a measure μ on $(X,\mathcal{S})$ by

$$\mu(E) = \sum_{x\in E} w(x)$$

for $E \subset X$. Prove that if $f\colon X \to [0,\infty]$ is a function, then

$$\int f\,d\mu = \sum_{x\in X} w(x)f(x),$$

where the infinite sums above are defined as the supremum of all sums over finite subsets of X.

8 Suppose λ denotes Lebesgue measure on $\mathbf{R}$. Given an example of a sequence $f_1, f_2, \dots$ of simple Borel measurable functions from $\mathbf{R}$ to $[0,\infty)$ such that $\lim_{k\to\infty} f_k(x) = 0$ for every $x \in \mathbf{R}$ but $\lim_{k\to\infty} \int f_k\,d\lambda = 1$.

9 Suppose μ is a measure on a measurable space $(X,\mathcal{S})$ and $f\colon X \to [0,\infty]$ is an $\mathcal{S}$-measurable function. Define $\nu\colon \mathcal{S} \to [0,\infty]$ by

$$\nu(A) = \int \chi_A f\,d\mu$$

for $A \in \mathcal{S}$. Prove that ν is a measure on $(X,\mathcal{S})$.

10 Suppose $(X,\mathcal{S},\mu)$ is a measure space and $f_1, f_2, \dots$ is a sequence of nonnegative $\mathcal{S}$-measurable functions. Define $f\colon X \to [0,\infty]$ by $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Prove that

$$\int f\,d\mu = \sum_{k=1}^{\infty} \int f_k\,d\mu.$$

11 Suppose $(X,\mathcal{S},\mu)$ is a measure space and $f_1, f_2, \dots$ are $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ such that $\sum_{k=1}^{\infty} \int |f_k|\,d\mu < \infty$. Prove that there exists $E \in \mathcal{S}$ such that $\mu(X \setminus E) = 0$ and $\lim_{k\to\infty} f_k(x) = 0$ for every $x \in E$.

12 Show that there exists a Borel measurable function $f\colon \mathbf{R} \to (0, \infty)$ such that $\int \chi_I f \, d\lambda = \infty$ for every nonempty open interval $I \subset \mathbf{R}$, where λ denotes Lebesgue measure on $\mathbf{R}$.

13 Give an example to show that the Monotone Convergence Theorem (3.11) can fail if the hypothesis that $f_1, f_2, \ldots$ are nonnegative functions is dropped.

14 Give an example to show that the Monotone Convergence Theorem can fail if the hypothesis of an increasing sequence of functions is replaced by a hypothesis of a decreasing sequence of functions.
[*This exercise shows that the Monotone Convergence Theorem should be called the Increasing Convergence Theorem. However, see Exercise 20.*]

15 Suppose λ is Lebesgue measure on $\mathbf{R}$ and $f\colon \mathbf{R} \to [-\infty, \infty]$ is a Borel measurable function such that $\int f \, d\lambda$ is defined.

(a) For $t \in \mathbf{R}$, define $f_t\colon \mathbf{R} \to [-\infty, \infty]$ by $f_t(x) = f(x - t)$. Prove that $\int f_t \, d\lambda = \int f \, d\lambda$ for all $t \in \mathbf{R}$

(b) For $t \in \mathbf{R} \setminus \{0\}$, define $f_t\colon \mathbf{R} \to [-\infty, \infty]$ by $f_t(x) = f(tx)$. Prove that $\int f_t \, d\lambda = \frac{1}{|t|} \int f \, d\lambda$ for all $t > 0$.

16 Suppose $\mathcal{S}$ and $\mathcal{T}$ are σ-algebras on a set X and $\mathcal{S} \subset \mathcal{T}$. Suppose μ_1 is a measure on $(X, \mathcal{S})$, μ_2 is a measure on $(X, \mathcal{T})$, and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{S}$. Prove that if $f\colon X \to [0, \infty]$ is $\mathcal{S}$-measurable, then $\int f \, d\mu_1 = \int f \, d\mu_2$.

For $x_1, x_2, \ldots$ a sequence in $[-\infty, \infty]$, define $\liminf_{k \to \infty} x_k$ by

$$\liminf_{k \to \infty} x_k = \lim_{k \to \infty} \inf\{x_k, x_{k+1}, \ldots\}.$$

Note that $\inf\{x_k, x_{k+1}, \ldots\}$ is an increasing function of k; thus the limit above on the right exists in $[-\infty, \infty]$.

17 Suppose that $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of nonnegative $\mathcal{S}$-measurable functions on X. Define a function $f\colon X \to [0, \infty]$ by $f(x) = \liminf_{k \to \infty} f_k(x)$.

(a) Show that f is an $\mathcal{S}$-measurable function.

(b) Prove that

$$\int f \, d\mu \leq \liminf_{k \to \infty} \int f_k \, d\mu.$$

(c) Give an example showing that the inequality in (b) can be a strict inequality even when $\mu(X) < \infty$ and the family of functions $\{f_k\}_{k \in \mathbf{Z}^+}$ is uniformly bounded.

[*The result in (b) is called Fatou's Lemma. Some textbooks prove Fatou's Lemma and then use it to prove the Monotone Convergence Theorem. Here we are taking the reverse approach—you should be able to use the Monotone Convergence Theorem to give a clean proof of Fatou's Lemma.*]

18 Give an example of a sequence $x_1, x_2, \ldots$ of real numbers such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k \text{ exists in } \mathbf{R},$$

but $\int x\,d\mu$ is not defined, where μ is counting measure on $\mathbf{Z}^+$ and x is the function from $\mathbf{Z}^+$ to $\mathbf{R}$ defined by $x(k) = x_k$.

19 Show that if $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [0, \infty)$ is $\mathcal{S}$-measurable, then

$$\mu(X) \inf_X f \leq \int f\,d\mu \leq \mu(X) \sup_X f.$$

20 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a monotone (meaning either increasing or decreasing) sequence of $\mathcal{S}$-measurable functions. Define $f\colon X \to [-\infty, \infty]$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Prove that if $\int |f_1|\,d\mu < \infty$, then

$$\lim_{k\to\infty} \int f_k\,d\mu = \int f\,d\mu.$$

21 Henri Lebesgue wrote the following about his method of integration:

> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

Use 3.15 to explain what Lebesgue meant and to explain why integration of a function with respect to a measure can be thought of as partitioning the range of the function, in contrast to Riemann integration, which depends upon partitioning the domain of the function.
[*The quote above is taken from page 796 of* The Princeton Companion to Mathematics, *edited by Timothy Gowers.*]

3B Limits of Integrals & Integrals of Limits

This section focuses on interchanging limits and integrals. Those tools allow us to characterize the Riemann integrable functions in terms of Lebesgue measure. We also develop some good approximation tools that will be useful in later chapters.

Bounded Convergence Theorem

We begin this section by introducing some useful notation.

3.24 Definition ***integration on a subset***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E \in \mathcal{S}$. If $f: X \to [-\infty, \infty]$ is an $\mathcal{S}$-measurable function, then $\int_E f \, d\mu$ is defined by

$$\int_E f \, d\mu = \int \chi_E f \, d\mu$$

if the right side of the equation above is defined; otherwise $\int_E f \, d\mu$ is undefined.

Alternatively, you can think of $\int_E f \, d\mu$ as $\int f|_E \, d\mu_E$, where μ_E is the measure obtained by restricting μ to the elements of $\mathcal{S}$ that are contained in E.

Notice that according to the definition above, the notation $\int_X f \, d\mu$ means the same as $\int f \, d\mu$. The following easy result illustrates the use of this new notation.

3.25 ***bounding an integral***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $E \in \mathcal{S}$, and $f: X \to [-\infty, \infty]$ is a function such that $\int_E f \, d\mu$ is defined. Then

$$\left| \int_E f \, d\mu \right| \leq \mu(E) \sup_E |f|.$$

Proof Let $c = \sup_E |f|$. We have

$$\begin{aligned} \left| \int_E f \, d\mu \right| &= \left| \int \chi_E f \, d\mu \right| \\ &\leq \int \chi_E |f| \, d\mu \\ &\leq \int c \chi_E \, d\mu \\ &= c\mu(E), \end{aligned}$$

where the second line comes from 3.23, the third line comes from 3.8, and the fourth line comes from 3.15. ■

The next result could be proved as a special case of the Dominated Convergence Theorem (3.31), which we prove later in this section. Thus you could skip the proof here. However, sometimes you get more insight by seeing an easier proof of an important special case. Thus you may want to read the easy proof of the Bounded Convergence Theorem that is presented next.

3.26 *Bounded Convergence Theorem*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ that converges pointwise on X to a function $f\colon X \to \mathbf{R}$. If there exists $c \in (0, \infty)$ such that

$$|f_k(x)| \leq c$$

for all $k \in \mathbf{Z}^+$ and all $x \in X$, then

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

Note the key role of Egorov's Theorem, which states that pointwise convergence is close to uniform convergence, in proofs involving interchanging limits and integrals.

Proof The function f is $\mathcal{S}$-measurable by 2.48.

Suppose c satisfies the hypothesis of this theorem. Let $\varepsilon > 0$. By Egorov's Theorem (2.85), there exists $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \frac{\varepsilon}{4c}$ and $f_1, f_2, \dots$ converges uniformly to f on E. Now

$$\begin{aligned}
\left|\int f_k \, d\mu - \int f \, d\mu\right| &= \left|\int_{X\setminus E} f_k \, d\mu - \int_{X\setminus E} f \, d\mu + \int_E (f_k - f) \, d\mu\right| \\
&\leq \int_{X\setminus E} |f_k| \, d\mu + \int_{X\setminus E} |f| \, d\mu + \int_E |f_k - f| \, d\mu \\
&< \frac{\varepsilon}{2} + \mu(E) \sup_E |f_k - f|,
\end{aligned}$$

where the last inequality follows from 3.25. Because $f_1, f_2, \dots$ converges uniformly to f on E and $\mu(E) < \infty$, the right side of the inequality above is less than ε for k sufficiently large, which completes the proof. ■

Sets of Measure 0 in Integration Theorems

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. If $f, g\colon X \to [-\infty, \infty]$ are $\mathcal{S}$-measurable functions and

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0,$$

then the definition of an integral implies that $\int f \, d\mu = \int g \, d\mu$ (or both integrals are undefined). Because what happens on a set of measure 0 often does not matter, the following definition is useful.

3.27 Definition ***almost every***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. A set $E \in \mathcal{S}$ is said to contain *μ-almost every* element of X if $\mu(X \setminus E) = 0$. If the measure μ is clear from the context, then the phrase *almost every* can be used (abbreviated by some authors to *a. e.*).

For example, almost every real number is irrational (with respect to the usual Lebesgue measure on $\mathbf{R}$) because $|\mathbf{Q}| = 0$.

Theorems about integrals can almost always be relaxed so that the hypotheses apply only almost everywhere instead of everywhere. For example, consider the Bounded Convergence Theorem (3.26), one of whose hypotheses is that

$$\lim_{k\to\infty} f_k(x) = f(x)$$

for all $x \in X$. Suppose that the hypotheses of the Bounded Convergence Theorem hold except that the equation above holds only almost everywhere, meaning there is a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) = 0$ and the equation above holds for all $x \in E$. Define new functions $g_1, g_2, \ldots$ and g by

$$g_k(x) = \begin{cases} f_k(x) & \text{if } x \in E, \\ 0 & \text{if } x \in X \setminus E \end{cases} \quad \text{and} \quad g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in X \setminus E. \end{cases}$$

Then

$$\lim_{k\to\infty} g_k(x) = g(x)$$

for all $x \in X$. Hence the Bounded Convergence Theorem implies that

$$\lim_{k\to\infty} \int g_k \, d\mu = \int g \, d\mu,$$

which immediately implies that

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$$

because $\int g_k \, d\mu = \int f_k \, d\mu$ and $\int g \, d\mu = \int f \, d\mu$.

Dominated Convergence Theorem

The next result tells us that if a nonnegative function has a finite integral, then its integral over all small sets (in the sense of measure) is small.

3.28 ***integrals on small sets are small***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $g \colon X \to [0, \infty]$ is $\mathcal{S}$-measurable, and $\int g \, d\mu < \infty$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_B g \, d\mu < \varepsilon$$

for every set $B \in \mathcal{S}$ such that $\mu(B) < \delta$.

Proof Suppose $\varepsilon > 0$. Let $h\colon X \to [0, \infty)$ be a simple $\mathcal{S}$-measurable function such that $0 \le h \le g$ and

$$\int g\,d\mu - \int h\,d\mu < \frac{\varepsilon}{2};$$

the existence of a function h with these properties follows from 3.9. Let

$$H = \max\{h(x) : x \in X\}$$

and let $\delta > 0$ be such that $H\delta < \frac{\varepsilon}{2}$.

Suppose $B \in \mathcal{S}$ and $\mu(B) < \delta$. Then

$$\begin{aligned}\int_B g\,d\mu &= \int_B (g-h)\,d\mu + \int_B h\,d\mu \\ &\le \int (g-h)\,d\mu + H\mu(B) \\ &< \frac{\varepsilon}{2} + H\delta \\ &< \varepsilon,\end{aligned}$$

as desired. ■

Some theorems, such as Egorov's Theorem (2.85) have as a hypothesis that the measure of the entire space is finite. The next result sometimes allows us to get around this hypothesis by restricting attention to a key set of finite measure.

3.29 ***integrable functions live mostly on sets of finite measure***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $g\colon X \to [0, \infty]$ is $\mathcal{S}$-measurable, and $\int g\,d\mu < \infty$. Then for every $\varepsilon > 0$, there exists $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} g\,d\mu < \varepsilon.$$

Proof Suppose $\varepsilon > 0$. Let P be an $\mathcal{S}$-partition $A_1, \dots, A_m$ of X such that

3.30 $$\int g\,d\mu < \varepsilon + \mathcal{L}(g, P).$$

Let E be the union of those A_j such that $\inf_{A_j} f > 0$. Then $\mu(E) < \infty$ (because otherwise we would have $\mathcal{L}(g, P) = \infty$, which contradicts the hypothesis that $\int g\,d\mu < \infty$). Now

$$\begin{aligned}\int_{X\setminus E} g\,d\mu &= \int g\,d\mu - \int \chi_E g\,d\mu \\ &< \bigl(\varepsilon + \mathcal{L}(g, P)\bigr) - \mathcal{L}(\chi_E g, P) \\ &= \varepsilon,\end{aligned}$$

where the second line follows from 3.30 and the definition of the integral of a nonnegative function, and the last line holds because $\inf_{A_j} f = 0$ for each A_j not contained in E. ■

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions on X such that $\lim_{k\to\infty} f_k(x) = f(x)$ for every (or almost every) $x \in X$. In general, it is not true that $\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$ (see Exercises 1 and 2).

We already have two good theorems about interchanging limits and integrals. However, both of these theorems have restrictive hypotheses. Specifically, the Monotone Convergence Theorem (3.11) requires all the functions to be nonnegative and it requires the sequence of functions to be increasing. The Bounded Convergence Theorem (3.26) requires the measure of the whole space to be finite and it requires the sequence of functions to be uniformly bounded by a constant.

The next theorem is the grand result in this area. It does not require the sequence of functions to be nonnegative, it does not require the sequence of functions to be increasing, it does not require the measure of the whole space to be finite, and it does not require the sequence of functions to be uniformly bounded. All these hypotheses are replaced only by a requirement that the sequence of functions is pointwise bounded by a function with a finite integral.

Notice that the Bounded Convergence Theorem follows immediately from the result below (take g to be an appropriate constant function and use the hypothesis in the Bounded Convergence Theorem that $\mu(X) < \infty$).

3.31 ***Dominated Convergence Theorem***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f \colon X \to [-\infty, \infty]$ is $\mathcal{S}$-measurable, and $f_1, f_2, \ldots$ are $\mathcal{S}$-measurable functions from X to $[-\infty, \infty]$ such that

$$\lim_{k\to\infty} f_k(x) = f(x)$$

for almost every $x \in X$. If there exists an $\mathcal{S}$-measurable function $g \colon X \to [0, \infty]$ such that

$$\int g \, d\mu < \infty \quad \text{and} \quad |f_k(x)| \leq g(x)$$

for every $k \in \mathbf{Z}^+$ and almost every $x \in X$, then

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

Proof Suppose $g \colon X \to [0, \infty]$ satisfies the hypotheses of this theorem. If $E \in \mathcal{S}$, then

$$\begin{aligned}
\Big|\int f_k \, d\mu - \int f \, d\mu\Big| &= \Big|\int_{X\setminus E} f_k \, d\mu - \int_{X\setminus E} f \, d\mu + \int_E f_k \, d\mu - \int_E f \, d\mu\Big| \\
&\leq \Big|\int_{X\setminus E} f_k \, d\mu\Big| + \Big|\int_{X\setminus E} f \, d\mu\Big| + \Big|\int_E f_k \, d\mu - \int_E f \, d\mu\Big| \\
&\leq 2\int_{X\setminus E} g \, d\mu + \Big|\int_E f_k \, d\mu - \int_E f \, d\mu\Big|. \qquad (3.32)
\end{aligned}$$

Case 1: Suppose $\mu(X) < \infty$.

Let $\varepsilon > 0$. By 3.28, there exists $\delta > 0$ such that

3.33
$$\int_B g \, d\mu < \frac{\varepsilon}{4}$$

for every set $B \in \mathcal{S}$ such that $\mu(B) < \delta$. By Egorov's Theorem (2.85), there exists a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \delta$ and $f_1, f_2, \ldots$ converges uniformly to f on E. Now 3.32 and 3.33 imply that

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| < \frac{\varepsilon}{2} + \left| \int_E (f_k - f) \, d\mu \right|.$$

Because $f_1, f_2, \ldots$ converges uniformly to f on E and $\mu(E) < \infty$, the last term on the right is less than $\frac{\varepsilon}{2}$ for all sufficiently large k. Thus $\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu$, completing the proof of case 1.

Case 2: Suppose $\mu(X) = \infty$.

Let $\varepsilon > 0$. By 3.29, there exists $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} g \, d\mu < \frac{\varepsilon}{4}.$$

The inequality above and 3.32 imply that

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| < \frac{\varepsilon}{2} + \left| \int_E f_k \, d\mu - \int_E f \, d\mu \right|.$$

By case 1 as applied to the sequence $f_1|_E, f_2|_E, \ldots$, the last term on the right is less than $\frac{\varepsilon}{2}$ for all sufficiently large k. Thus $\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu$, completing the proof of case 2. ∎

Riemann Integrals and Lebesgue Integrals

We can now use the tools we have developed to characterize the Riemann integrable functions. In the theorem below, the left side of the last equation denotes the Riemann integral.

3.34 *Riemann integrable $\iff$ continuous almost everywhere*

Suppose $a < b$ and $f \colon [a, b] \to \mathbf{R}$ is a bounded function. Then f is Riemann integrable if and only if

$$|\{x \in [a, b] : f \text{ is not continuous at } x\}| = 0.$$

Furthermore, if f is Riemann integrable and λ denotes Lebesgue measure on $\mathbf{R}$, then f is Lebesgue measurable and

$$\int_a^b f = \int_{[a,b]} f \, d\lambda.$$

Proof Suppose $n \in \mathbf{Z}^+$. Consider the partition P_n that divides $[a,b]$ into 2^n subintervals of equal size. Let $I_1, \dots, I_{2^n}$ be the corresponding closed subintervals, each of length $(b-a)/2^n$. Let

3.35 $$g_n = \sum_{j=1}^{2^n} (\inf_{I_j} f)\chi_{I_j} \quad \text{and} \quad h_n = \sum_{j=1}^{2^n} (\sup_{I_j} f)\chi_{I_j}.$$

The lower and upper Riemann sums of f for the partition P_n are given by integrals. Specifically,

3.36 $$L(f, P_n, [a,b]) = \int_{[a,b]} g_n \, d\lambda \quad \text{and} \quad U(f, P_n, [a,b]) = \int_{[a,b]} h_n \, d\lambda,$$

where λ is Lebesgue measure on $\mathbf{R}$.

The definitions of g_n and h_n given in 3.35 are actually just a first draft of the definitions. A slight problem arises at each point that is in two of the intervals $I_1, \dots, I_{2^n}$ (in other words, at endpoints of these intervals other than a and b). At each of these points, change the value of g_n to be the infimum of f over the union of the two intervals that contain the point, and change the value of h_n to be the supremum of f over the union of the two intervals that contain the point. This change modifies g_n and h_n on only a finite number of points. Thus the integrals in 3.36 are not affected. This change is needed in order to make 3.38 true (otherwise the two sets in 3.38 might differ by at most countably many points, which would not really change the proof but which would not be as aesthetically pleasing).

Clearly $g_1 \le g_2 \le \cdots$ is an increasing sequence of functions and $h_1 \ge h_2 \ge \cdots$ is a decreasing sequence of functions on $[a,b]$. Define functions $f^{\mathrm{L}} \colon [a,b] \to \mathbf{R}$ and $f^{\mathrm{U}} \colon [a,b] \to \mathbf{R}$ by

$$f^{\mathrm{L}}(x) = \lim_{n\to\infty} g_n(x) \quad \text{and} \quad f^{\mathrm{U}}(x) = \lim_{n\to\infty} h_n(x).$$

Taking the limit as $n \to \infty$ of both equations in 3.36 and using the Bounded Convergence Theorem (3.26) along with Exercise 7 in Section 1A, we see that f^L and f^U are Lebesgue measurable functions and

3.37 $$L(f, [a,b]) = \int_{[a,b]} f^{\mathrm{L}} \, d\lambda \quad \text{and} \quad U(f, [a,b]) = \int_{[a,b]} f^{\mathrm{U}} \, d\lambda.$$

Now 3.37 implies that f is Riemann integrable if and only if

$$\int_{[a,b]} (f^{\mathrm{U}} - f^{\mathrm{L}}) \, d\lambda = 0.$$

Because $f^{\mathrm{L}}(x) \le f(x) \le f^{\mathrm{U}}(x)$ for all $x \in [a,b]$, the equation above holds if and only if

$$|\{x \in [a,b] : f^{\mathrm{U}}(x) \ne f^{\mathrm{L}}(x)\}| = 0.$$

The remaining details of the proof can be completed by noting that

3.38 $$\{x \in [a,b] : f^{\mathrm{U}}(x) \ne f^{\mathrm{L}}(x)\} = \{x \in [a,b] : f \text{ is not continuous at } x\}.$$ ∎

We previously defined the notation $\int_a^b f$ to mean the Riemann integral of f. Because the Riemann integral and Lebesgue integral agree for Riemann integrable functions (see 3.34), we now redefine $\int_a^b f$ to denote the Lebesgue integral.

3.39 Definition $\int_a^b f$

Suppose $-\infty \le a < b \le \infty$ and $f\colon (a,b) \to \mathbf{R}$ is Lebesgue measurable. Then

- $\int_a^b f$ and $\int_a^b f(x)\,dx$ mean $\int_{(a,b)} f\,d\lambda$, where λ is Lebesgue measure on $\mathbf{R}$;
- $\int_b^a f$ is defined to be $-\int_a^b f$.

The definition in the second bullet point above is made so that equations such as

$$\int_a^b f = \int_a^c f + \int_c^b f$$

remain valid even if, for example, $a < b < c$.

Approximation by Nice Functions

In the next definition, the notation $\|f\|_1$ should be $\|f\|_{1,\mu}$ because it depends upon the measure μ as well as upon f. However, μ is usually clear from the context. In some books, you may see the notation $\mathcal{L}^1(X,\mathcal{S},\mu)$ instead of $\mathcal{L}^1(\mu)$.

3.40 Definition $\|f\|_1$; $\mathcal{L}^1(\mu)$

Suppose $(X,\mathcal{S},\mu)$ is a measure space. If $f\colon X \to [-\infty,\infty]$ is $\mathcal{S}$-measurable, then the $\mathcal{L}^1$-*norm* of f is denoted by $\|f\|_1$ and is defined by

$$\|f\|_1 = \int |f|\,d\mu.$$

The *Lebesgue space* $\mathcal{L}^1(\mu)$ is defined by

$$\mathcal{L}^1(\mu) = \{f : f \text{ is an } \mathcal{S}\text{-measurable function from } X \text{ to } \mathbf{R} \text{ and } \|f\|_1 < \infty\}.$$

The terminology and notation used above are convenient even though $\|\cdot\|_1$ might not be a genuine norm (to be defined in Chapter 6).

3.41 Example $\mathcal{L}^1(\mu)$ *functions that take on only finitely many values*

Suppose $(X,\mathcal{S},\mu)$ is a measure space and $E_1,\dots,E_n$ are disjoint subsets of X. Suppose $a_1,\dots,a_n$ are distinct nonzero real numbers. Then

$$a_1\chi_{E_1} + \cdots + a_n\chi_{E_n} \in \mathcal{L}^1(\mu)$$

if and only if $E_k \in \mathcal{S}$ and $\mu(E_k) < \infty$ for all $k \in \{1,\dots,n\}$. Furthermore,

$$\|a_1\chi_{E_1} + \cdots + a_n\chi_{E_n}\|_1 = |a_1|\mu(E_1) + \cdots + |a_n|\mu(E_n).$$

3.42 Example ℓ^1

If μ is counting measure on $\mathbf{Z}^+$ and $x = x_1, x_2, \dots$ is a sequence of real numbers (thought of as a function on $\mathbf{Z}^+$), then $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$. In this case, $\mathcal{L}^1(\mu)$ is often denoted by ℓ^1 (pronounced *little-el-one*). In other words, ℓ^1 is the set of all sequences $x_1, x_2, \dots$ of real numbers such that $\sum_{k=1}^{\infty} |x_k| < \infty$.

The easy proof of the following result is left to the reader.

3.43 *properties of the $\mathcal{L}^1$-norm*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, g \in \mathcal{L}^1(\mu)$. Then

- $\|f\|_1 \geq 0$;
- $\|f\|_1 = 0$ if and only if $f(x) = 0$ for almost every $x \in X$;
- $\|cf\|_1 = |c|\|f\|_1$ for all $c \in \mathbf{R}$;
- $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

The next result states that every function in $\mathcal{L}^1(\mu)$ can be approximated in $\mathcal{L}^1$-norm by measurable functions that take on only finitely many values.

3.44 *approximation by simple functions*

Suppose μ is a measure and $f \in \mathcal{L}^1(\mu)$. Then for every $\varepsilon > 0$, there exists a simple function $g \in \mathcal{L}^1(\mu)$ such that

$$\|f - g\|_1 < \varepsilon.$$

Proof Suppose $\varepsilon > 0$. Then there exist simple functions $g_1, g_2 \in \mathcal{L}^1(\mu)$ such that $0 \leq g_1 \leq f^+$ and $0 \leq g_2 \leq f^-$ and

$$\int (f^+ - g_1)\, d\mu < \frac{\varepsilon}{2} \quad \text{and} \quad \int (f^- - g_2)\, d\mu < \frac{\varepsilon}{2},$$

where we have used 3.9 to provide the existence of g_1, g_2 with these properties.

Let $g = g_1 - g_2$. Then g is a simple function in $\mathcal{L}^1(\mu)$ and

$$\begin{aligned}
\|f - g\|_1 &= \|(f^+ - g_1) - (f^- - g_2)\|_1 \\
&= \int (f^+ - g_1)\, d\mu + \int (f^- - g_2)\, d\mu \\
&< \varepsilon,
\end{aligned}$$

as desired. ∎

3.45 Definition $\mathcal{L}^1(\mathbf{R})$; $\|f\|_1$

- The notation $\mathcal{L}^1(\mathbf{R})$ denotes $\mathcal{L}^1(\lambda)$, where λ is Lebesgue measure on either the Borel subsets of $\mathbf{R}$ or the Lebesgue measurable subsets of $\mathbf{R}$.
- When working with $\mathcal{L}^1(\mathbf{R})$, the notation $\|f\|_1$ denotes the integral of the absolute value of f with respect to Lebesgue measure on $\mathbf{R}$.

3.46 Definition *step function*

A *step function* is a function $g\colon \mathbf{R} \to \mathbf{R}$ of the form

$$g = a_1\chi_{I_1} + \cdots + a_n\chi_{I_n},$$

where $I_1, \ldots, I_n$ are intervals of $\mathbf{R}$ and $a_1, \ldots, a_n$ are nonzero real numbers.

Suppose g is a step function of the form above and the intervals $I_1, \ldots, I_n$ are disjoint. Then

$$\|g\|_1 = |a_1|\,|I_1| + \cdots + |a_n|\,|I_n|.$$

In particular, $g \in \mathcal{L}^1(\mathbf{R})$ if and only if all the intervals $I_1, \ldots, I_n$ are bounded.

The intervals in the definition of a step function can be open intervals, closed intervals, or half-open intervals. We will be using step functions in integrals, where the inclusion or exclusion of the endpoints of the intervals does not matter.

Even though the coefficients $a_1, \ldots, a_n$ in the definition of a step function are required to be nonzero, the function 0 that is identically 0 on $\mathbf{R}$ is a step function. To see this, take $n = 1$, $a_1 = 1$, and $I_1 = \varnothing$.

3.47 ***approximation by step functions***

Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Then for every $\varepsilon > 0$, there exists a step function $g \in \mathcal{L}^1(\mathbf{R})$ such that

$$\|f - g\|_1 < \varepsilon.$$

Proof Suppose $\varepsilon > 0$. By 3.44, there exist Borel (or Lebesgue) measurable subsets $A_1, \ldots, A_n$ of $\mathbf{R}$ and nonzero numbers $a_1, \ldots, a_n$ such that $|A_k| < \infty$ for all $k \in \{1, \ldots, n\}$ and

$$\Big\| f - \sum_{k=1}^{n} a_k\chi_{A_k} \Big\|_1 < \frac{\varepsilon}{2}.$$

For each $k \in \{1, \ldots, n\}$, there is an open subset G_k of $\mathbf{R}$ that contains A_k and whose Lebesgue measure is as close as we want to $|A_k|$ [by part (e) of 2.71]. Each open subset of $\mathbf{R}$, including each G_k, is a countable union of disjoint open intervals. Thus for each k, there is a set E_k that is a finite union of bounded open intervals contained in G_k whose Lebesgue measure is as close as we want to $|G_k|$. Hence for each k, there is a set E_k that is a finite union of bounded intervals such that

$$|E_k \setminus A_k| + |A_k \setminus E_k| \le |G_k \setminus A_k| + |G_k \setminus E_k|$$
$$< \frac{\varepsilon}{2|a_k|n};$$

in other words,

$$\|\chi_{A_k} - \chi_{E_k}\|_1 < \frac{\varepsilon}{2|a_k|n}.$$

Now

$$\Big\|f - \sum_{k=1}^{n} a_k\chi_{E_k}\Big\|_1 \le \Big\|f - \sum_{k=1}^{n} a_k\chi_{A_k}\Big\|_1 + \Big\|\sum_{k=1}^{n} a_k\chi_{A_k} - \sum_{k=1}^{n} a_k\chi_{E_k}\Big\|_1$$
$$< \frac{\varepsilon}{2} + \sum_{k=1}^{n} |a_k|\|\chi_{A_k} - \chi_{E_k}\|_1$$
$$< \varepsilon.$$

Each E_k is a finite union of bounded intervals. Thus the inequality above completes the proof because $\sum_{k=1}^{n} a_k\chi_{E_k}$ is a step function. ■

Luzin's Theorem (2.91 and 2.93) gives a spectacular way to approximate a Borel measurable function by a continuous function. However, the following approximation theorem is usually more useful than Luzin's Theorem. For example, the next result plays a major role in the proof of the Lebesgue Differentiation Theorem (4.10).

3.48 ***approximation by continuous functions***

Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Then for every $\varepsilon > 0$, there exists a continuous function $g \colon \mathbf{R} \to \mathbf{R}$ such that

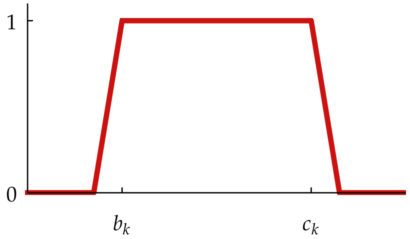
$$\|f - g\|_1 < \varepsilon$$

and $\{x \in \mathbf{R} : g(x) \neq 0\}$ is a bounded set.

Proof For every $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n \in \mathbf{R}$ and $g_1, \dots, g_n \in \mathcal{L}^1(\mathbf{R})$, we have

$$\Big\|f - \sum_{k=1}^{n} a_k g_k\Big\|_1 \le \Big\|f - \sum_{k=1}^{n} a_k\chi_{[b_k, c_k]}\Big\|_1 + \Big\|\sum_{k=1}^{n} a_k(\chi_{[b_k, c_k]} - g_k)\Big\|_1$$
$$\le \Big\|f - \sum_{k=1}^{n} a_k\chi_{[b_k, c_k]}\Big\|_1 + \sum_{k=1}^{n} |a_k|\,\|\chi_{[b_k, c_k]} - g_k\|_1,$$

where the inequalities above follow from 3.43. By 3.47, we can choose $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n \in \mathbf{R}$ to make $\|f - \sum_{k=1}^{n} a_k\chi_{[b_k, c_k]}\|_1$ as small as we wish. The figure here then shows that there exist continuous functions $g_1, \dots, g_n \in \mathcal{L}^1(\mathbf{R})$ that make $\sum_{k=1}^{n} |a_k|\|\chi_{[b_k, c_k]} - g_k\|_1$ as small as we wish. Now take $g = \sum_{k=1}^{n} a_k g_k$. ■



The graph of a continuous function g_k such that $\|\chi_{[b_k, c_k]} - g_k\|_1$ is small.

EXERCISES 3B

1 Give an example of a sequence $f_1, f_2, \ldots$ of functions from $\mathbf{Z}^+$ to $[0, \infty)$ such that

$$\lim_{k\to\infty} f_k(m) = 0$$

for every $m \in \mathbf{Z}^+$ but $\lim_{k\to\infty} \int f_k \, d\mu = 1$, where μ is counting measure on $\mathbf{Z}^+$.

2 Give an example of a sequence $f_1, f_2, \ldots$ of continuous functions from $\mathbf{R}$ to $[0, 1]$ such that

$$\lim_{k\to\infty} f_k(x) = 0$$

for every $x \in \mathbf{R}$ but $\lim_{k\to\infty} \int f_k \, d\lambda = \infty$, where λ is Lebesgue measure on $\mathbf{R}$.

3 Suppose λ is Lebesgue measure on $\mathbf{R}$ and $f \colon \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that $\int |f| \, d\lambda < \infty$. Define $g \colon \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \int_{(-\infty, x)} f \, d\lambda.$$

Prove that g is uniformly continuous on $\mathbf{R}$.

4 (a) Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose that $f \colon X \to [0, \infty)$ is a bounded $\mathcal{S}$-measurable function. Prove that

$$\int f \, d\mu = \inf\Bigl\{ \sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A_1, \ldots, A_m \text{ is an } \mathcal{S}\text{-partition of } X \Bigr\}.$$

(b) Show that the conclusion of part (a) can fail if the hypothesis that f is bounded is replaced by the hypothesis that $\int f \, d\mu < \infty$.

(c) Show that the conclusion of part (a) can fail if the condition that $\mu(X) < \infty$ is deleted.

[*Part* (*a*) *of this exercise shows that if we had defined an upper Lebesgue sum, then we could have used it to define the integral. However, parts* (*b*) *and* (*c*) *show that the hypotheses that f is bounded and that $\mu(X) < \infty$ would be needed if defining the integral via the equation above. The definition of the integral via the lower Lebesgue sum does not require these hypotheses, showing the advantage of using the approach via the lower Lebesgue sum.*]

5 Let λ denote Lebesgue measure on $\mathbf{R}$. Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that $\int |f| \, d\lambda < \infty$. Prove that

$$\lim_{k\to\infty} \int_{[-k, k]} f \, d\lambda = \int f \, d\lambda.$$

6 Let λ denote Lebesgue measure on $\mathbf{R}$. Give an example of a continuous function $f \colon [0, \infty) \to \mathbf{R}$ such that $\lim_{t\to\infty} \int_{[0, t]} f \, d\lambda$ exists (in $\mathbf{R}$) but $\int_{[0, \infty)} f \, d\lambda$ is not defined.

7 Let λ denote Lebesgue measure on $\mathbf{R}$. Give an example of a continuous function $f\colon (0,1) \to \mathbf{R}$ such that $\lim_{n\to\infty} \int_{(\frac{1}{n},1)} f\, d\lambda$ exists (in $\mathbf{R}$) but $\int_{(0,1)} f\, d\lambda$ is not defined.

8 Verify the assertion in 3.38.

9 Verify the assertion in Example 3.41.

10 (a) Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$. Suppose p, r are positive numbers with $p < r$. Prove that if $f\colon X \to [0, \infty)$ is an $\mathcal{S}$-measurable function such that $\int f^r\, d\mu < \infty$, then $\int f^p\, d\mu < \infty$.

(b) Give an example to show that the result in part (a) can be false without the hypothesis that $\mu(X) < \infty$.

11 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \in \mathcal{L}^1(\mu)$. Prove that

$$\{x \in X : f(x) \neq 0\}$$

is the countable union of sets with finite μ-measure.

12 Suppose

$$f_k(x) = \frac{(1-x)^k \cos\frac{k}{x}}{\sqrt{x}}.$$

Prove that $\displaystyle\lim_{k\to\infty} \int_0^1 f_k = 0$.

13 Give an example of a sequence of nonnegative Borel measurable functions $f_1, f_2, \dots$ on $[0,1]$ such that both the following conditions hold:

- $\displaystyle\lim_{k\to\infty} \int_0^1 f_k = 0$;
- $\displaystyle\sup_{k\ge m} f_k(x) = \infty$ for every $m \in \mathbf{Z}^+$ and every $x \in [0,1]$.

14 Let λ denote Lebesgue measure on $\mathbf{R}$.

(a) Let $f(x) = 1/\sqrt{x}$. Prove that $\int_{[0,1]} f\, d\lambda = 2$.

(b) Let $f(x) = 1/(1+x^2)$. Prove that $\int_{\mathbf{R}} f\, d\lambda = \pi$.

(c) Let $f(x) = (\sin x)/x$. Show that the integral $\int_{(0,\infty)} f\, d\lambda$ is not defined but $\lim_{t\to\infty} \int_{(0,t)} f\, d\lambda$ exists in $\mathbf{R}$.

15 Prove or give a counterexample: If G is an open subset of $(0,1)$, then χ_G is Riemann integrable on $[0,1]$.

16 Suppose $f \in \mathcal{L}^1(\mathbf{R})$.

(a) For $t \in \mathbf{R}$, define $f_t\colon \mathbf{R} \to \mathbf{R}$ by $f_t(x) = f(x-t)$. Prove that

$$\lim_{t\to 0} \|f - f_t\|_1 = 0.$$

(b) For $t > 0$, define $f_t\colon \mathbf{R} \to \mathbf{R}$ by $f_t(x) = f(tx)$. Prove that

$$\lim_{t\to 1} \|f - f_t\|_1 = 0.$$

Chapter 4
Differentiation

Does there exist a Lebesgue measurable set that fills up exactly half of each interval? To get a feeling for this question, consider the set $E = [0, \frac{1}{8}] \cup [\frac{1}{4}, \frac{3}{8}] \cup [\frac{1}{2}, \frac{5}{8}] \cup [\frac{3}{4}, \frac{7}{8}]$. This set E has the property that

$$|E \cap [0, b]| = \frac{b}{2}$$

for $b = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. Does there exist a Lebesgue measurable set $E \subset [0, 1]$, perhaps constructed in a fashion similar to the Cantor set, such that the equation above holds for all $b \in [0, 1]$?

In this chapter we see how to answer this question by considering differentiation issues. We begin by developing a powerful tool called the Hardy–Littlewood maximal inequality. This tool is used to prove an almost everywhere version of the Fundamental Theorem of Calculus. These results lead us to an important theorem about the density of Lebesgue measurable sets.



Trinity College at the University of Cambridge in England. G. H. Hardy (1877–1947) and John Littlewood (1885–1977) were students and later faculty members here. If you have not already done so, you should read Hardy's remarkable book A Mathematician's Apology *(do not skip the fascinating Foreword by C. P. Snow) and see the movie* The Man Who Knew Infinity*, which focuses on Hardy, Littlewood, and Srinivasa Ramanujan (1887–1920).*

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_4

4A Hardy–Littlewood Maximal Function

Markov's Inequality

The following result, called Markov's inequality, has a sweet, short proof. We will make good use of this result later in this chapter (see the proof of 4.10). Markov's inequality also leads to Chebyshev's inequality (see Exercise 2 in this section).

4.1 *Markov's inequality*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h \in \mathcal{L}^1(\mu)$. Then

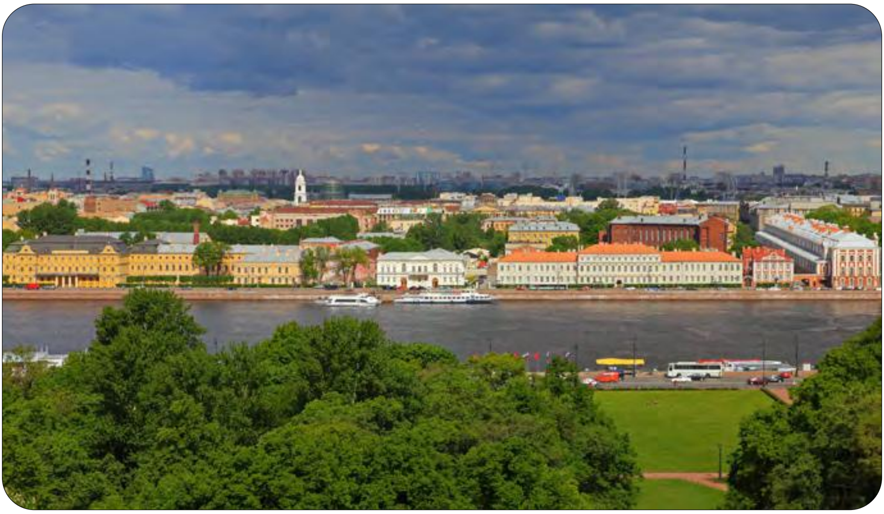
$$\mu(\{x \in X : |h(x)| \geq c\}) \leq \frac{1}{c}\|h\|_1$$

for every $c > 0$.

Proof Suppose $c > 0$. Then

$$\begin{aligned} \mu(\{x \in X : |h(x)| \geq c\}) &= \frac{1}{c}\int_{\{x \in X : |h(x)| \geq c\}} c \, d\mu \\ &\leq \frac{1}{c}\int_{\{x \in X : |h(x)| \geq c\}} |h| \, d\mu \\ &\leq \frac{1}{c}\|h\|_1, \end{aligned}$$

as desired. ■



St. Petersburg University along the Neva River in St. Petersburg, Russia. Andrei Markov (1856–1922) was a student and then a faculty member here.

Vitali Covering Lemma

4.2 Definition *3 times a bounded nonempty open interval*

Suppose I is a bounded nonempty open interval of $\mathbf{R}$. Then $3 * I$ denotes the open interval with the same center as I and three times the length of I.

4.3 Example *3 times an interval*

If $I = (0, 10)$, then $3 * I = (-10, 20)$.

The next result is a key tool in the proof of the Hardy–Littlewood maximal inequality (4.8).

4.4 ***Vitali Covering Lemma***

Suppose $I_1, \dots, I_n$ is a list of bounded nonempty open intervals of $\mathbf{R}$. Then there exists a disjoint sublist $I_{k_1}, \dots, I_{k_m}$ such that

$$I_1 \cup \cdots \cup I_n \subset (3 * I_{k_1}) \cup \cdots \cup (3 * I_{k_m}).$$

4.5 Example *Vitali Covering Lemma*

Suppose $n = 4$ and

$$I_1 = (0, 10), \quad I_2 = (9, 15), \quad I_3 = (14, 22), \quad I_4 = (21, 31).$$

Then

$$3 * I_1 = (-10, 20), \quad 3 * I_2 = (3, 21), \quad 3 * I_3 = (6, 30), \quad 3 * I_4 = (11, 41).$$

Thus

$$I_1 \cup I_2 \cup I_3 \cup I_4 \subset (3 * I_1) \cup (3 * I_4).$$

In this example, I_1, I_4 is the only sublist of I_1, I_2, I_3, I_4 that produces the conclusion of the Vitali Covering Lemma.

Proof of 4.4 Let k_1 be such that

$$|I_{k_1}| = \max\{|I_1|, \dots, |I_n|\}.$$

Suppose $k_1, \dots, k_j$ have been chosen. Let k_{j+1} be such that $|I_{k_{j+1}}|$ is as large as possible subject to the condition that $I_{k_1}, \dots, I_{k_{j+1}}$ are disjoint. If there is no choice of k_{j+1} such that $I_{k_1}, \dots, I_{k_{j+1}}$ are disjoint, then the procedure terminates. Because we start with a finite list, the procedure must eventually terminate after some number m of choices.

The technique used here is called a greedy algorithm because at each stage we select the largest remaining interval that is disjoint from the previously selected intervals.

Suppose $j \in \{1, \dots, n\}$. To complete the proof, we must show that

$$I_j \subset (3 * I_{k_1}) \cup \cdots \cup (3 * I_{k_m}).$$

If $j \in \{k_1, \dots, k_m\}$, then the inclusion above obviously holds.

Thus assume that $j \notin \{k_1, \dots, k_m\}$. Because the process terminated without selecting j, the interval I_j is not disjoint from all of $I_{k_1}, \dots, I_{k_m}$. Let I_{k_L} be the first interval on this list not disjoint from I_j; thus I_j is disjoint from $I_{k_1}, \dots, I_{k_{L-1}}$. Because j was not chosen in step L, we conclude that $|I_{k_L}| \geq |I_j|$. Because $I_{k_L} \cap I_j \neq \emptyset$, this last inequality implies (easy exercise) that $I_j \subset 3 * I_{k_L}$, completing the proof. ■

Hardy–Littlewood Maximal Inequality

Now we come to a brilliant definition that turns out to be extraordinarily useful.

4.6 Definition ***Hardy–Littlewood maximal function***

Suppose $h \colon \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function. Then the *Hardy–Littlewood maximal function* of h is the function $h^* \colon \mathbf{R} \to [0, \infty]$ defined by

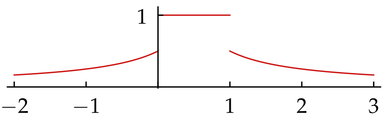
$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} |h|.$$

In other words, $h^*(b)$ is the supremum over all bounded intervals centered at b of the average of $|h|$ on those intervals.

4.7 Example *Hardy–Littlewood maximal function of $\chi_{[0,1]}$*

As usual, let $\chi_{[0,1]}$ denote the characteristic function of the interval $[0,1]$. Then

$$(\chi_{[0,1]})^*(b) = \begin{cases} \frac{1}{2(1-b)} & \text{if } b \leq 0, \\ 1 & \text{if } 0 < b < 1, \\ \frac{1}{2b} & \text{if } b \geq 1, \end{cases}$$



The graph of $(\chi_{[0,1]})^*$ on $[-2,3]$.

as you should verify.

If $h \colon \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable and $c \in \mathbf{R}$, then $\{b \in \mathbf{R} : h^*(b) > c\}$ is an open subset of $\mathbf{R}$, as you are asked to prove in Exercise 9 in this section. Thus h^* is a Borel measurable function.

Suppose $h \in \mathcal{L}^1(\mathbf{R})$ and $c > 0$. Markov's inequality (4.1) estimates the size of the set on which $|h|$ is larger than c. Our next result estimates the size of the set on which h^* is larger than c. The Hardy–Littlewood maximal inequality proved in the next result is a key ingredient in the proof of the Lebesgue Differentiation Theorem (4.10). Note that this next result is considerably deeper than Markov's inequality.

4.8 *Hardy–Littlewood maximal inequality*

Suppose $h \in \mathcal{L}^1(\mathbf{R})$. Then

$$|\{b \in \mathbf{R} : h^*(b) > c\}| \leq \frac{3}{c}\|h\|_1$$

for every $c > 0$.

Proof Suppose F is a closed bounded subset of $\{b \in \mathbf{R} : h^*(b) > c\}$. We will show that $|F| \leq \frac{3}{c}\int_{-\infty}^{\infty}|h|$, which implies our desired result [see Exercise 24(a) in Section 2D].

For each $b \in F$, there exists $t_b > 0$ such that

4.9
$$\frac{1}{2t_b}\int_{b-t_b}^{b+t_b}|h| > c.$$

Clearly

$$F \subset \bigcup_{b \in F}(b - t_b, b + t_b).$$

The Heine–Borel Theorem (2.12) tells us that this open cover of a closed bounded set has a finite subcover. In other words, there exist $b_1, \dots, b_n \in F$ such that

$$F \subset (b_1 - t_{b_1}, b_1 + t_{b_1}) \cup \cdots \cup (b_n - t_{b_n}, b_n + t_{b_n}).$$

To make the notation cleaner, relabel the open intervals above as $I_1, \dots, I_n$.

Now apply the Vitali Covering Lemma (4.4) to the list $I_1, \dots, I_n$, producing a disjoint sublist $I_{k_1}, \dots, I_{k_m}$ such that

$$I_1 \cup \cdots \cup I_n \subset (3 * I_{k_1}) \cup \cdots \cup (3 * I_{k_m}).$$

Thus

$$\begin{aligned}
|F| &\leq |I_1 \cup \cdots \cup I_n| \\
&\leq |(3 * I_{k_1}) \cup \cdots \cup (3 * I_{k_m})| \\
&\leq |3 * I_{k_1}| + \cdots + |3 * I_{k_m}| \\
&= 3(|I_{k_1}| + \cdots + |I_{k_m}|) \\
&< \frac{3}{c}\Big(\int_{I_{k_1}}|h| + \cdots + \int_{I_{k_m}}|h|\Big) \\
&\leq \frac{3}{c}\int_{-\infty}^{\infty}|h|,
\end{aligned}$$

where the second-to-last inequality above comes from 4.9 (note that $|I_{k_j}| = 2t_b$ for the choice of b corresponding to I_{k_j}) and the last inequality holds because $I_{k_1}, \dots, I_{k_m}$ are disjoint.

The last inequality completes the proof. ■

EXERCISES 4A

1 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h\colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function. Prove that

$$\mu(\{x \in X : |h(x)| \geq c\}) \leq \frac{1}{c^p} \int |h|^p \, d\mu$$

for all positive numbers c and p.

2 Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) = 1$ and $h \in \mathcal{L}^1(\mu)$. Prove that

$$\mu\Bigl(\Bigl\{x \in X : \Bigl|h(x) - \int h \, d\mu\Bigr| \geq c\Bigr\}\Bigr) \leq \frac{1}{c^2}\Bigl(\int h^2 \, d\mu - \Bigl(\int h \, d\mu\Bigr)^2\Bigr)$$

for all $c > 0$.

[*The result above is called Chebyshev's inequality; it plays an important role in probability theory. Pafnuty Chebyshev* (*1821–1894*) *was Markov's thesis advisor.*]

3 Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Suppose $h \in \mathcal{L}^1(\mu)$ and $\|h\|_1 > 0$. Prove that there is at most one number $c \in (0, \infty)$ such that

$$\mu(\{x \in X : |h(x)| \geq c\}) = \frac{1}{c}\|h\|_1.$$

4 Show that the constant 3 in the Vitali Covering Lemma (4.4) cannot be replaced by a smaller positive constant.

5 Prove the assertion left as an exercise in the last sentence of the proof of the Vitali Covering Lemma (4.4).

6 Verify the formula in Example 4.7 for the Hardy–Littlewood maximal function of $\chi_{[0,\,1]}$.

7 Find a formula for the Hardy–Littlewood maximal function of the characteristic function of $[0, 1] \cup [2, 3]$.

8 Find a formula for the Hardy–Littlewood maximal function of the function $h\colon \mathbf{R} \to [0, \infty)$ defined by

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

9 Suppose $h\colon \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable. Prove that

$$\{b \in \mathbf{R} : h^*(b) > c\}$$

is an open subset of $\mathbf{R}$ for every $c \in \mathbf{R}$.

10 Prove or give a counterexample: If $h\colon \mathbf{R} \to [0,\infty)$ is an increasing function, then h^* is an increasing function.

11 Give an example of a Borel measurable function $h\colon \mathbf{R} \to [0,\infty)$ such that $h^*(b) < \infty$ for all $b \in \mathbf{R}$ but $\sup\{h^*(b) : b \in \mathbf{R}\} = \infty$.

12 Show that $|\{b \in \mathbf{R} : h^*(b) = \infty\}| = 0$ for every $h \in \mathcal{L}^1(\mathbf{R})$.

13 Show that there exists $h \in \mathcal{L}^1(\mathbf{R})$ such that $h^*(b) = \infty$ for every $b \in \mathbf{Q}$.

14 Suppose $h \in \mathcal{L}^1(\mathbf{R})$. Prove that

$$|\{b \in \mathbf{R} : h^*(b) \geq c\}| \leq \frac{3}{c}\|h\|_1$$

for every $c > 0$.

[*This result slightly strengthens the Hardy–Littlewood maximal inequality* (*4.8*) *because the set on the left side above includes those* $b \in \mathbf{R}$ *such that* $h^*(b) = c$. *A much deeper strengthening comes from replacing the constant* 3 *in the Hardy–Littlewood maximal inequality with a smaller constant. In 2003, Antonios Melas answered what had been an open question about the best constant. He proved that the smallest constant that can replace* 3 *in the Hardy–Littlewood maximal inequality is* $(11 + \sqrt{61})/12 \approx 1.56752$; *see* Annals of Mathematics *157* (*2003*), *647–688.*]

4B Derivatives of Integrals

Lebesgue Differentiation Theorem

The next result states that the average amount by which a function in $\mathcal{L}^1(\mathbf{R})$ differs from its values is small almost everywhere on small intervals. The 2 in the denominator of the fraction in the result below could be deleted, but its presence makes the length of the interval of integration nicely match the denominator $2t$.

The next result is called the Lebesgue Differentiation Theorem, even though no derivative is in sight. However, we will soon see how another version of this result deals with derivatives. The hard work takes place in the proof of this first version.

4.10 *Lebesgue Differentiation Theorem, first version*

Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Then

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| = 0$$

for almost every $b \in \mathbf{R}$.

Before getting to the formal proof of this first version of the Lebesgue Differentiation Theorem, we pause to provide some motivation for the proof. If $b \in \mathbf{R}$ and $t > 0$, then 3.25 gives the easy estimate

$$\frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| \leq \sup\{|f(x) - f(b)| : |x - b| \leq t\}.$$

If f is continuous at b, then the right side of this inequality has limit 0 as $t \downarrow 0$, proving 4.10 in the special case in which f is continuous on $\mathbf{R}$.

To prove the Lebesgue Differentiation Theorem, we will approximate an arbitrary function in $\mathcal{L}^1(\mathbf{R})$ by a continuous function (using 3.48). The previous paragraph shows that the continuous function has the desired behavior. We will use the Hardy–Littlewood maximal inequality (4.8) to show that the approximation produces approximately the desired behavior. Now we are ready for the formal details of the proof.

Proof of 4.10 Let $\delta > 0$. By 3.48, for each $k \in \mathbf{Z}^+$ there exists a continuous function $h_k \colon \mathbf{R} \to \mathbf{R}$ such that

4.11
$$\|f - h_k\|_1 < \frac{\delta}{k2^k}.$$

Let

$$B_k = \{b \in \mathbf{R} : |f(b) - h_k(b)| \leq \tfrac{1}{k} \text{ and } (f - h_k)^*(b) \leq \tfrac{1}{k}\}.$$

Then

4.12
$$\mathbf{R} \setminus B_k = \{b \in \mathbf{R} : |f(b) - h_k(b)| > \tfrac{1}{k}\} \cup \{b \in \mathbf{R} : (f - h_k)^*(b) > \tfrac{1}{k}\}.$$

Markov's inequality (4.1) as applied to the function $f - h_k$ and 4.11 imply that

4.13 $$|\{b \in \mathbf{R} : |f(b) - h_k(b)| > \tfrac{1}{k}\}| < \frac{\delta}{2^k}.$$

The Hardy–Littlewood maximal inequality (4.8) as applied to the function $f - h_k$ and 4.11 imply that

4.14 $$|\{b \in \mathbf{R} : (f - h_k)^*(b) > \tfrac{1}{k}\}| < \frac{3\delta}{2^k}.$$

Now 4.12, 4.13, and 4.14 imply that

$$|\mathbf{R} \setminus B_k| < \frac{\delta}{2^{k-2}}.$$

Let

$$B = \bigcap_{k=1}^{\infty} B_k.$$

Then

4.15 $$|\mathbf{R} \setminus B| = \Bigl|\bigcup_{k=1}^{\infty} (\mathbf{R} \setminus B_k)\Bigr| \leq \sum_{k=1}^{\infty} |\mathbf{R} \setminus B_k| < \sum_{k=1}^{\infty} \frac{\delta}{2^{k-2}} = 4\delta.$$

Suppose $b \in B$ and $t > 0$. Then for each $k \in \mathbf{Z}^+$ we have

$$\begin{aligned}
\frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| &\leq \frac{1}{2t} \int_{b-t}^{b+t} \bigl(|f - h_k| + |h_k - h_k(b)| + |h_k(b) - f(b)|\bigr) \\
&\leq (f - h_k)^*(b) + \Bigl(\frac{1}{2t} \int_{b-t}^{b+t} |h_k - h_k(b)|\Bigr) + |h_k(b) - f(b)| \\
&\leq \frac{2}{k} + \frac{1}{2t} \int_{b-t}^{b+t} |h_k - h_k(b)|.
\end{aligned}$$

Because h_k is continuous, the last term is less than $\frac{1}{k}$ for all $t > 0$ sufficiently close to 0 (how close is sufficiently close depends upon k). In other words, for each $k \in \mathbf{Z}^+$, we have

$$\frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| < \frac{3}{k}$$

for all $t > 0$ sufficiently close to 0.

Hence we conclude that

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| = 0$$

for all $b \in B$.

Let A denote the set of numbers $a \in \mathbf{R}$ such that

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{a-t}^{a+t} |f - f(a)|$$

either does not exist or is nonzero. We have shown that $A \subset (\mathbf{R} \setminus B)$. Thus

$$|A| \leq |\mathbf{R} \setminus B| < 4\delta,$$

where the last inequality comes from 4.15. Because δ is an arbitrary positive number, the last inequality implies that $|A| = 0$, completing the proof. ∎

Derivatives

You should remember the following definition from your calculus course.

4.16 Definition ***derivative***

Suppose $g\colon I \to \mathbf{R}$ is a function defined on an open interval I of $\mathbf{R}$ and $b \in I$. The *derivative* of g at b, denoted $g'(b)$, is defined by

$$g'(b) = \lim_{t \to 0} \frac{g(b+t) - g(b)}{t}$$

if the limit above exists, in which case g is called *differentiable* at b.

We now turn to the Fundamental Theorem of Calculus and a powerful extension that avoids continuity. These results show that differentiation and integration can be thought of as inverse operations.

You saw the next result in your calculus class, except now the function f is only required to be Lebesgue measurable (and its absolute value must have a finite Lebesgue integral). Of course, we also need to require f to be continuous at the crucial point b in the next result, because changing the value of f at a single number would not change the function g.

4.17 ***Fundamental Theorem of Calculus***

Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Define $g\colon \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \int_{-\infty}^{x} f.$$

Suppose $b \in \mathbf{R}$ and f is continuous at b. Then g is differentiable at b and

$$g'(b) = f(b).$$

Proof If $t \neq 0$, then

$$\begin{aligned} \left| \frac{g(b+t) - g(b)}{t} - f(b) \right| &= \left| \frac{\int_{-\infty}^{b+t} f - \int_{-\infty}^{b} f}{t} - f(b) \right| \\ &= \left| \frac{\int_{b}^{b+t} f}{t} - f(b) \right| \\ &= \left| \frac{\int_{b}^{b+t} \bigl(f - f(b)\bigr)}{t} \right| \qquad (4.18) \\ &\leq \sup_{\{x \in \mathbf{R} \colon |x-b| < |t|\}} |f(x) - f(b)|. \end{aligned}$$

If $\varepsilon > 0$, then by the continuity of f at b, the last quantity is less than ε for t sufficiently close to 0. Thus g is differentiable at b and $g'(b) = f(b)$. ■

A function in $\mathcal{L}^1(\mathbf{R})$ need not be continuous anywhere. Thus the Fundamental Theorem of Calculus (4.17) might provide no information about differentiating the integral of such a function. However, our next result states that all is well almost everywhere, even in the absence of any continuity of the function being integrated.

4.19 *Lebesgue Differentiation Theorem, second version*

Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Define $g\colon \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \int_{-\infty}^{x} f.$$

Then $g'(b) = f(b)$ for almost every $b \in \mathbf{R}$.

Proof Suppose $t \neq 0$. Then from 4.18 we have

$$\begin{aligned}\left|\frac{g(b+t)-g(b)}{t} - f(b)\right| &= \left|\frac{\int_b^{b+t}\bigl(f - f(b)\bigr)}{t}\right| \\ &\leq \frac{1}{t}\int_b^{b+t} |f - f(b)| \\ &\leq \frac{1}{t}\int_{b-t}^{b+t} |f - f(b)|\end{aligned}$$

for all $b \in \mathbf{R}$. By the first version of the Lebesgue Differentiation Theorem (4.10), the last quantity has limit 0 as $t \to 0$ for almost every $b \in \mathbf{R}$. Thus $g'(b) = f(b)$ for almost every $b \in \mathbf{R}$. ■

Now we can answer the question raised on the opening page of this chapter.

4.20 *no set constitutes exactly half of each interval*

There does not exist a Lebesgue measurable set $E \subset [0,1]$ such that

$$|E \cap [0,b]| = \frac{b}{2}$$

for all $b \in [0,1]$.

Proof Suppose there does exist a Lebesgue measurable set $E \subset [0,1]$ with the property above. Define $g\colon \mathbf{R} \to \mathbf{R}$ by

$$g(b) = \int_{-\infty}^{b} \chi_E\,.$$

Thus $g(b) = \frac{b}{2}$ for all $b \in [0,1]$. Hence $g'(b) = \frac{1}{2}$ for all $b \in (0,1)$.

The Lebesgue Differentiation Theorem (4.19) implies that $g'(b) = \chi_E(b)$ for almost every $b \in \mathbf{R}$. However, χ_E never takes on the value $\frac{1}{2}$, which contradicts the conclusion of the previous paragraph. This contradiction completes the proof. ■

The next result says that a function in $\mathcal{L}^1(\mathbf{R})$ is equal almost everywhere to the limit of its average over small intervals. These two-sided results generalize more naturally to higher dimensions (take the average over balls centered at b) than the one-sided results.

4.21 ***$\mathcal{L}^1(\mathbf{R})$ function equals its local average almost everywhere***

Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Then

$$f(b) = \lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} f$$

for almost every $b \in \mathbf{R}$.

Proof Suppose $t > 0$. Then

$$\begin{aligned} \left| \left(\frac{1}{2t} \int_{b-t}^{b+t} f \right) - f(b) \right| &= \left| \frac{1}{2t} \int_{b-t}^{b+t} \bigl(f - f(b)\bigr) \right| \\ &\leq \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|. \end{aligned}$$

The desired result now follows from the first version of the Lebesgue Differentiation Theorem (4.10). ■

Again, the conclusion of the result above holds at every number b at which f is continuous. The remarkable part of the result above is that even if f is discontinuous everywhere, the conclusion holds for almost every real number b.

Density

The next definition captures the notion of the proportion of a set in small intervals centered at a number b.

4.22 Definition ***density***

Suppose $E \subset \mathbf{R}$. The *density* of E at a number $b \in \mathbf{R}$ is

$$\lim_{t \downarrow 0} \frac{|E \cap (b-t, b+t)|}{2t}$$

if this limit exists (otherwise the density of E at b is undefined).

4.23 Example *density of an interval*

$$\text{The density of } [0,1] \text{ at } b = \begin{cases} 1 & \text{if } b \in (0,1), \\ \frac{1}{2} & \text{if } b = 0 \text{ or } b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The next beautiful result shows the power of the techniques developed in this chapter.

4.24 *Lebesgue Density Theorem*

Suppose $E \subset \mathbf{R}$ is a Lebesgue measurable set. Then the density of E is 1 at almost every element of E and is 0 at almost every element of $\mathbf{R} \setminus E$.

Proof First suppose $|E| < \infty$. Thus $\chi_E \in \mathcal{L}^1(\mathbf{R})$. Because

$$\frac{|E \cap (b-t, b+t)|}{2t} = \frac{1}{2t} \int_{b-t}^{b+t} \chi_E$$

for every $t > 0$ and every $b \in \mathbf{R}$, the desired result follows immediately from 4.21.

Now consider the case where $|E| = \infty$ [which means that $\chi_E \notin \mathcal{L}^1(\mathbf{R})$ and hence 4.21 as stated cannot be used]. For $k \in \mathbf{Z}^+$, let $E_k = E \cap (-k, k)$. If $|b| < k$, then the density of E at b equals the density of E_k at b. By the previous paragraph as applied to E_k, there are sets $F_k \subset E_k$ and $G_k \subset \mathbf{R} \setminus E_k$ such that $|F_k| = |G_k| = 0$ and the density of E_k equals 1 at every element of $E_k \setminus F_k$ and the density of E_k equals 0 at every element of $(\mathbf{R} \setminus E_k) \setminus G_k$.

Let $F = \bigcup_{k=1}^{\infty} F_k$ and $G = \bigcup_{k=1}^{\infty} G_k$. Then $|F| = |G| = 0$ and the density of E is 1 at every element of $E \setminus F$ and is 0 at every element of $(\mathbf{R} \setminus E) \setminus G$. ∎

The bad Borel set provided by the next result leads to a bad Borel measurable function. Specifically, let E be the bad Borel set in 4.25. Then χ_E is a Borel measurable function that is discontinuous everywhere. Furthermore, the function χ_E cannot be modified on a set of measure 0 to be continuous anywhere (in contrast to the function $\chi_{\mathbf{Q}}$).

The Lebesgue Density Theorem makes the example provided by the next result somewhat surprising. Be sure to spend some time pondering why the next result does not contradict the Lebesgue Density Theorem. Also, compare the next result to 4.20.

Even though the function χ_E discussed in the paragraph above is continuous nowhere and every modification of this function on a set of measure 0 is also continuous nowhere, the function g defined by

$$g(b) = \int_0^b \chi_E$$

is differentiable almost everywhere (by 4.19).

The proof of 4.25 given below is based on an idea of Walter Rudin.

4.25 *bad Borel set*

There exists a Borel set $E \subset \mathbf{R}$ such that

$$0 < |E \cap I| < |I|$$

for every nonempty bounded open interval I.

Proof We use the following fact in our construction:

4.26 Suppose G is a nonempty open subset of $\mathbf{R}$. Then there exists a closed set $F \subset G \setminus \mathbf{Q}$ such that $|F| > 0$.

To prove 4.26, let J be a closed interval contained in G such that $0 < |J|$. Let $r_1, r_2, \ldots$ be a list of all the rational numbers. Let

$$F = J \setminus \bigcup_{k=1}^{\infty} \Bigl(r_k - \frac{|J|}{2^{k+2}}, r_k + \frac{|J|}{2^{k+2}}\Bigr).$$

Then F is a closed subset of $\mathbf{R}$ and $F \subset J \setminus \mathbf{Q} \subset G \setminus \mathbf{Q}$. Also, $|J \setminus F| \leq \frac{1}{2}|J|$ because $J \setminus F \subset \bigcup_{k=1}^{\infty} \Bigl(r_k - \frac{|J|}{2^{k+2}}, r_k + \frac{|J|}{2^{k+2}}\Bigr)$. Thus

$$|F| = |J| - |J \setminus F| \geq \tfrac{1}{2}|J| > 0,$$

completing the proof of 4.26.

To construct the set E with the desired properties, let $I_1, I_2, \ldots$ be a sequence consisting of all nonempty bounded open intervals of $\mathbf{R}$ with rational endpoints. Let $F_0 = \hat{F}_0 = \varnothing$, and inductively construct sequences $F_1, F_2, \ldots$ and $\hat{F}_1, \hat{F}_2, \ldots$ of closed subsets of $\mathbf{R}$ as follows: Suppose $n \in \mathbf{Z}^+$ and $F_0, \ldots, F_{n-1}$ and $\hat{F}_0, \ldots, \hat{F}_{n-1}$ have been chosen as closed sets that contain no rational numbers. Thus

$$I_n \setminus (\hat{F}_0 \cup \ldots \cup \hat{F}_{n-1})$$

is a nonempty open set (nonempty because it contains all rational numbers in I_n). Applying 4.26 to the open set above, we see that there is a closed set F_n contained in the set above such that F_n contains no rational numbers and $|F_n| > 0$. Applying 4.26 again, but this time to the open set

$$I_n \setminus (F_0 \cup \ldots \cup F_n),$$

which is nonempty because it contains all rational numbers in I_n, we see that there is a closed set $\hat{F}_n$ contained in the set above such that $\hat{F}_n$ contains no rational numbers and $|\hat{F}_n| > 0$.

Now let

$$E = \bigcup_{k=1}^{\infty} F_k.$$

Our construction implies that $F_k \cap \hat{F}_n = \varnothing$ for all $k, n \in \mathbf{Z}^+$. Thus $E \cap \hat{F}_n = \varnothing$ for all $n \in \mathbf{Z}^+$. Hence $\hat{F}_n \subset I_n \setminus E$ for all $n \in \mathbf{Z}^+$.

Suppose I is a nonempty bounded open interval. Then $I_n \subset I$ for some $n \in \mathbf{Z}^+$. Thus

$$0 < |F_n| \leq |E \cap I_n| \leq |E \cap I|.$$

Also,

$$|E \cap I| = |I| - |I \setminus E| \leq |I| - |I_n \setminus E| \leq |I| - |\hat{F}_n| < |I|,$$

completing the proof. ∎

EXERCISES 4B

For $f \in \mathcal{L}^1(\mathbf{R})$ and I an interval of $\mathbf{R}$ with $0 < |I| < \infty$, let f_I denote the average of f on I. In other words, $f_I = \frac{1}{|I|} \int_I f$.

1 Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Prove that

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f_{[b-t,\, b+t]}| = 0$$

for almost every $b \in \mathbf{R}$.

2 Suppose $f \in \mathcal{L}^1(\mathbf{R})$. Prove that

$$\lim_{t \downarrow 0} \sup \biggl\{ \frac{1}{|I|} \int_I |f - f_I| : I \text{ is an interval of length } t \text{ containing } b \biggr\} = 0$$

for almost every $b \in \mathbf{R}$.

3 Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function such that $f^2 \in \mathcal{L}^1(\mathbf{R})$. Prove that

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = 0$$

for almost every $b \in \mathbf{R}$.

4 Prove that the Lebesgue Differentiation Theorem (4.19) still holds if the hypothesis that $\int_{-\infty}^{\infty} |f| < \infty$ is weakened to the requirement that $\int_{-\infty}^{x} |f| < \infty$ for all $x \in \mathbf{R}$.

5 Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function. Prove that

$$|f(b)| \leq f^*(b)$$

for almost every $b \in \mathbf{R}$.

6 Prove that if $h \in L^1(\mathbf{R})$ and $\int_{-\infty}^{s} h = 0$ for all $s \in \mathbf{R}$, then $h = 0$.

7 Give an example of a Borel subset of $\mathbf{R}$ whose density at 0 is not defined.

8 Give an example of a Borel subset of $\mathbf{R}$ whose density at 0 is $\frac{1}{3}$.

9 Prove that if $t \in [0, 1]$, then there exists a Borel set $E \subset \mathbf{R}$ such that the density of E at 0 is t.

10 Suppose E is a Lebesgue measurable subset of $\mathbf{R}$ such that the density of E equals 1 at every element of E and equals 0 at every element of $\mathbf{R} \setminus E$. Prove that $E = \varnothing$ or $E = \mathbf{R}$.

Chapter 5
Product Measures

Lebesgue measure on $\mathbf{R}$ generalizes the notion of the length of an interval. In this chapter, we see how two-dimensional Lebesgue measure on $\mathbf{R}^2$ generalizes the notion of the area of a rectangle. More generally, we construct new measures that are the products of two measures.

Once these new measures have been constructed, the question arises of how to compute integrals with respect to these new measures. Beautiful theorems proved in the first decade of the twentieth century allow us to compute integrals with respect to product measures as iterated integrals involving the two measures that produced the product. Furthermore, we will see that under reasonable conditions we can switch the order of an iterated integral.



Main building of Scuola Normale Superiore di Pisa, the university in Pisa, Italy, where Guido Fubini (1879–1943) received his PhD in 1900. In 1907 Fubini proved that under reasonable conditions, an integral with respect to a product measure can be computed as an iterated integral and that the order of integration can be switched. Leonida Tonelli (1885–1943) also taught for many years in Pisa; he also proved a crucial theorem about interchanging the order of integration in an iterated integral.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_5

5A Products of Measure Spaces

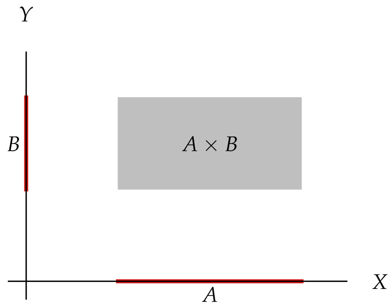
Products of σ-Algebras

Our first step in constructing product measures is to construct the product of two σ-algebras. We begin with the following definition.

5.1 Definition ***rectangle***

Suppose X and Y are sets. A *rectangle* in $X \times Y$ is a set of the form $A \times B$, where $A \subset X$ and $B \subset Y$.

Keep the figure shown here in mind when thinking of a rectangle in the sense defined above. However, remember that A and B need not be intervals as shown in the figure. Indeed, the concept of an interval makes no sense in the generality of arbitrary sets.



Now we can define the product of two σ-algebras.

5.2 Definition ***product of two σ-algebras; $\mathcal{S} \otimes \mathcal{T}$***

Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Then

- the *product* $\mathcal{S} \otimes \mathcal{T}$ is defined to be the smallest σ-algebra on $X \times Y$ that contains
$$\{A \times B : A \in \mathcal{S}, B \in \mathcal{T}\};$$
- a *measurable rectangle* in $\mathcal{S} \otimes \mathcal{T}$ is a set of the form $A \times B$, where $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

Using the terminology introduced in the second bullet point above, we can say that $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra containing all the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Exercise 1 in this section asks you to show that the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ are the only rectangles in $X \times Y$ that are in $\mathcal{S} \otimes \mathcal{T}$.

The notation $\mathcal{S} \times \mathcal{T}$ is not used because $\mathcal{S}$ and $\mathcal{T}$ are sets (of sets), and thus the notation $\mathcal{S} \times \mathcal{T}$ already is defined to mean the set of all ordered pairs of the form (A, B), where $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

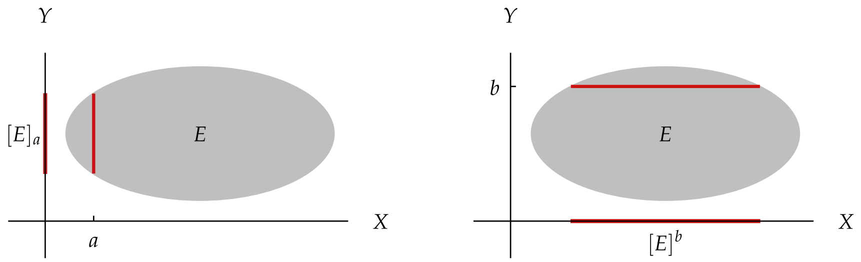
The notion of cross sections plays a crucial role in our development of product measures. First, we define cross sections of sets, and then we define cross sections of functions.

5.3 Definition ***cross sections of sets; $[E]_a$ and $[E]^b$***

Suppose X and Y are sets and $E \subset X \times Y$. Then for $a \in X$ and $b \in Y$, the *cross sections* $[E]_a$ and $[E]^b$ are defined by

$$[E]_a = \{y \in Y : (a, y) \in E\} \quad \text{and} \quad [E]^b = \{x \in X : (x, b) \in E\}.$$

5.4 Example *cross sections of a subset of $X \times Y$*



5.5 Example *cross sections of rectangles*

Suppose X and Y are sets and $A \subset X$ and $B \subset Y$. If $a \in X$ and $b \in Y$, then

$$[A \times B]_a = \begin{cases} B & \text{if } a \in A, \\ \varnothing & \text{if } a \notin A \end{cases} \quad \text{and} \quad [A \times B]^b = \begin{cases} A & \text{if } b \in B, \\ \varnothing & \text{if } b \notin B, \end{cases}$$

as you should verify.

The next result shows that cross sections preserve measurability.

5.6 ***cross sections of measurable sets are measurable***

Suppose $\mathcal{S}$ is a σ-algebra on X and $\mathcal{T}$ is a σ-algebra on Y. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

$$[E]_a \in \mathcal{T} \text{ for every } a \in X \quad \text{and} \quad [E]^b \in \mathcal{S} \text{ for every } b \in Y.$$

Proof Let $\mathcal{E}$ denote the collection of subsets E of $X \times Y$ for which the conclusion of this result holds. Then $A \times B \in \mathcal{E}$ for all $A \in \mathcal{S}$ and all $B \in \mathcal{T}$ (by Example 5.5).

The collection $\mathcal{E}$ is closed under complementation and countable unions because

$$[(X \times Y) \setminus E]_a = Y \setminus [E]_a$$

and

$$[E_1 \cup E_2 \cup \cdots]_a = [E_1]_a \cup [E_2]_a \cup \cdots$$

for all subsets $E, E_1, E_2, \ldots$ of $X \times Y$ and all $a \in X$, as you should verify, with similar statements holding for cross sections with respect to all $b \in Y$.

Because $\mathcal{E}$ is a σ-algebra containing all the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$, we conclude that $\mathcal{E}$ contains $\mathcal{S} \otimes \mathcal{T}$. ■

Now we define cross sections of functions.

5.7 Definition ***cross sections of functions; $[f]_a$ and $[f]^b$***

Suppose X and Y are sets and $f\colon X \times Y \to \mathbf{R}$ is a function. Then for $a \in X$ and $b \in Y$, the cross section functions $[f]_a\colon Y \to \mathbf{R}$ and $[f]^b\colon X \to \mathbf{R}$ are defined by

$$[f]_a(y) = f(a, y) \text{ for } y \in Y \quad \text{and} \quad [f]^b(x) = f(x, b) \text{ for } x \in X.$$

5.8 Example *cross sections*

- Suppose $f\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined by $f(x, y) = 5x^2 + y^3$. Then

  $$[f]_2(y) = 20 + y^3 \quad \text{and} \quad [f]^3(x) = 5x^2 + 27$$

  for all $y \in \mathbf{R}$ and all $x \in \mathbf{R}$, as you should verify.

- Suppose X and Y are sets and $A \subset X$ and $B \subset Y$. If $a \in X$ and $b \in Y$, then

  $$[\chi_{A \times B}]_a = \chi_A(a)\chi_B \quad \text{and} \quad [\chi_{A \times B}]^b = \chi_B(b)\chi_A,$$

  as you should verify.

The next result shows that cross sections preserve measurability, this time in the context of functions rather than sets.

5.9 ***cross sections of measurable functions are measurable***

Suppose $\mathcal{S}$ is a σ-algebra on X and $\mathcal{T}$ is a σ-algebra on Y. Suppose $f\colon X \times Y \to \mathbf{R}$ is an $\mathcal{S} \otimes \mathcal{T}$-measurable function. Then

$$[f]_a \text{ is a } \mathcal{T}\text{-measurable function on } Y \text{ for every } a \in X$$

and

$$[f]^b \text{ is an } \mathcal{S}\text{-measurable function on } X \text{ for every } b \in Y.$$

Proof Suppose D is a Borel subset of $\mathbf{R}$ and $a \in X$. If $y \in Y$, then

$$\begin{aligned} y \in ([f]_a)^{-1}(D) &\iff [f]_a(y) \in D \\ &\iff f(a, y) \in D \\ &\iff (a, y) \in f^{-1}(D) \\ &\iff y \in [f^{-1}(D)]_a. \end{aligned}$$

Thus

$$([f]_a)^{-1}(D) = [f^{-1}(D)]_a.$$

Because f is an $\mathcal{S} \otimes \mathcal{T}$-measurable function, $f^{-1}(D) \in \mathcal{S} \otimes \mathcal{T}$. Thus the equation above and 5.6 imply that $([f]_a)^{-1}(D) \in \mathcal{T}$. Hence $[f]_a$ is a $\mathcal{T}$-measurable function.

The same ideas show that $[f]^b$ is an $\mathcal{S}$-measurable function for every $b \in Y$. ■

Monotone Class Theorem

The following standard two-step technique often works to prove that every set in a σ-algebra has a certain property:

1. show that every set in a collection of sets that generates the σ-algebra has the property;

2. show that the collection of sets that has the property is a σ-algebra.

For example, the proof of 5.6 used the technique above—first we showed that every measurable rectangle in $\mathcal{S} \otimes \mathcal{T}$ has the desired property, then we showed that the collection of sets that has the desired property is a σ-algebra (this completed the proof because $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra containing the measurable rectangles).

The technique outlined above should be used when possible. However, in some situations there seems to be no reasonable way to verify that the collection of sets with the desired property is a σ-algebra. We will encounter this situation in the next subsection. To deal with it, we need to introduce another technique that involves what are called monotone classes.

The following definition will be used in our main theorem about monotone classes.

5.10 Definition ***algebra***

Suppose W is a set and $\mathcal{A}$ is a set of subsets of W. Then $\mathcal{A}$ is called an *algebra* on W if the following three conditions are satisfied:

- $\varnothing \in \mathcal{A}$;
- if $E \in \mathcal{A}$, then $W \setminus E \in \mathcal{A}$;
- if E and F are elements of $\mathcal{A}$, then $E \cup F \in \mathcal{A}$.

Thus an algebra is closed under complementation and under finite unions; a σ-algebra is closed under complementation and countable unions.

5.11 Example *collection of finite unions of intervals is an algebra*

Suppose $\mathcal{A}$ is the collection of all finite unions of intervals of $\mathbf{R}$. Here we are including all intervals—open intervals, closed intervals, bounded intervals, unbounded intervals, sets consisting of only a single point, and intervals that are neither open nor closed because they contain one endpoint but not the other endpoint.

Clearly $\mathcal{A}$ is closed under finite unions. You should also verify that $\mathcal{A}$ is closed under complementation. Thus $\mathcal{A}$ is an algebra on $\mathbf{R}$.

5.12 Example *collection of countable unions of intervals is not an algebra*

Suppose $\mathcal{A}$ is the collection of all countable unions of intervals of $\mathbf{R}$.

Clearly $\mathcal{A}$ is closed under finite unions (and also under countable unions). You should verify that $\mathcal{A}$ is not closed under complementation. Thus $\mathcal{A}$ is neither an algebra nor a σ-algebra on $\mathbf{R}$.

The following result provides an example of an algebra that we will exploit.

5.13 ***the set of finite unions of measurable rectangles is an algebra***

Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Then

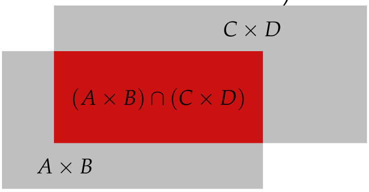
(a) the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ is an algebra on $X \times Y$;

(b) every finite union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ can be written as a finite union of disjoint measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$.

Proof Let $\mathcal{A}$ denote the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Obviously $\mathcal{A}$ is closed under finite unions.

The collection $\mathcal{A}$ is also closed under finite intersections. To verify this claim, note that if $A_1, \dots, A_n, C_1, \dots, C_m \in \mathcal{S}$ and $B_1, \dots, B_n, D_1, \dots, D_m \in \mathcal{T}$, then

$$\Big((A_1 \times B_1) \cup \cdots \cup (A_n \times B_n)\Big) \bigcap \Big((C_1 \times D_1) \cup \cdots \cup (C_m \times D_m)\Big)$$
$$= \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \Big((A_j \times B_j) \cap (C_k \times D_k)\Big)$$
$$= \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \Big((A_j \cap C_k) \times (B_j \cap D_k)\Big),$$



Intersection of two rectangles is a rectangle.

which implies that $\mathcal{A}$ is closed under finite intersections.

If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then

$$(X \times Y) \setminus (A \times B) = \Big((X \setminus A) \times Y\Big) \cup \Big(X \times (Y \setminus B)\Big).$$

Hence the complement of each measurable rectangle in $\mathcal{S} \otimes \mathcal{T}$ is in $\mathcal{A}$. Thus the complement of a finite union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ is in $\mathcal{A}$ (use De Morgan's Laws and the result in the previous paragraph that $\mathcal{A}$ is closed under finite intersections). In other words, $\mathcal{A}$ is closed under complementation, completing the proof of (a).

To prove (b), note that if $A \times B$ and $C \times D$ are measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$, then (as can be verified in the figure above)

5.14 $$(A \times B) \cup (C \times D) = (A \times B) \cup \Big(C \times (D \setminus B)\Big) \cup \Big((C \setminus A) \times (B \cap D)\Big).$$

The equation above writes the union of two measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ as the union of three disjoint measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$.

Now consider any finite union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. If this is not a disjoint union, then choose any nondisjoint pair of measurable rectangles in the union and replace those two measurable rectangles with the union of three disjoint measurable rectangles as in 5.14. Iterate this process until obtaining a disjoint union of measurable rectangles. ■

Now we define a monotone class as a collection of sets that is closed under countable increasing unions and under countable decreasing intersections.

5.15 Definition ***monotone class***

Suppose W is a set and $\mathcal{M}$ is a set of subsets of W. Then $\mathcal{M}$ is called a *monotone class* on W if the following two conditions are satisfied:

- If $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{M}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$;
- If $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{M}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.

Clearly every σ-algebra is a monotone class. However, some monotone classes are not closed under even finite unions, as shown by the next example.

5.16 Example *a monotone class that is not an algebra*

Suppose $\mathcal{A}$ is the collection of all intervals of $\mathbf{R}$. Then $\mathcal{A}$ is closed under countable increasing unions and countable decreasing intersections. Thus $\mathcal{A}$ is a monotone class on $\mathbf{R}$. However, $\mathcal{A}$ is not closed under finite unions, and $\mathcal{A}$ is not closed under complementation. Thus $\mathcal{A}$ is neither an algebra nor a σ-algebra on $\mathbf{R}$.

If $\mathcal{A}$ is a collection of subsets of some set W, then the intersection of all monotone classes on W that contain A is a monotone class that contains $\mathcal{A}$. Thus this intersection is the smallest monotone class on W that contains $\mathcal{A}$.

The next result provides a useful tool when the standard technique for showing that every set in a σ-algebra has a certain property does not work.

5.17 ***Monotone Class Theorem***

Suppose $\mathcal{A}$ is an algebra on a set W. Then the smallest σ-algebra containing $\mathcal{A}$ is the smallest monotone class containing $\mathcal{A}$.

Proof Let $\mathcal{M}$ denote the smallest monotone class containing $\mathcal{A}$. Because every σ-algebra is a monotone class, $\mathcal{M}$ is contained in the smallest σ-algebra containing $\mathcal{A}$.

To prove the inclusion in the other direction, first suppose $A \in \mathcal{A}$. Let

$$\mathcal{E} = \{E \in \mathcal{M} : A \cup E \in \mathcal{M}\}.$$

Then $\mathcal{A} \subset \mathcal{E}$ (because the union of two sets in $\mathcal{A}$ is in $\mathcal{A}$). A moment's thought shows that $\mathcal{E}$ is a monotone class. Thus the smallest monotone class that contains $\mathcal{A}$ is contained in $\mathcal{E}$, meaning that $\mathcal{M} \subset \mathcal{E}$. Hence we have proved that $A \cup E \in \mathcal{M}$ for every $E \in \mathcal{M}$.

Now let

$$\mathcal{D} = \{D \in \mathcal{M} : D \cup E \in \mathcal{M} \text{ for all } E \in \mathcal{M}\}.$$

The previous paragraph shows that $\mathcal{A} \subset \mathcal{D}$. A moment's thought again shows that $\mathcal{D}$ is a monotone class. Thus, as in the previous paragraph, we conclude that $\mathcal{M} \subset \mathcal{D}$. Hence we have proved that $D \cup E \in \mathcal{M}$ for all $D, E \in \mathcal{M}$.

The paragraph above shows that the monotone class $\mathcal{M}$ is closed under finite unions. Now if $E_1, E_2, \ldots \in \mathcal{M}$, then

$$E_1 \cup E_2 \cup E_3 \cup \cdots = E_1 \cup (E_1 \cup E_2) \cup (E_1 \cup E_2 \cup E_3) \cup \cdots ,$$

which is an increasing union of a sequence of sets in $\mathcal{M}$ (by the previous paragraph). We conclude that $\mathcal{M}$ is closed under countable unions.

Finally, let

$$\mathcal{M}' = \{E \in \mathcal{M} : W \setminus E \in \mathcal{M}\}.$$

Then $\mathcal{A} \subset \mathcal{M}'$ (because $\mathcal{A}$ is closed under complementation). Once again, you should verify that $\mathcal{M}'$ is a monotone class. Thus $\mathcal{M} \subset \mathcal{M}'$. We conclude that $\mathcal{M}$ is closed under complementation.

The two previous paragraphs show that $\mathcal{M}$ is closed under countable unions and under complementation. Thus $\mathcal{M}$ is a σ-algebra that contains $\mathcal{A}$. Hence $\mathcal{M}$ contains the smallest σ-algebra containing $\mathcal{A}$, completing the proof. ■

Products of Measures

The following definitions will be useful.

5.18 Definition ***finite measure; σ-finite measure***

- A measure μ on a measurable space $(X, \mathcal{S})$ is called *finite* if $\mu(X) < \infty$.
- A measure is called *σ-finite* if the whole space can be written as the countable union of sets with finite measure.
- More precisely, a measure μ on a measurable space $(X, \mathcal{S})$ is called *σ-finite* if there exists a sequence $X_1, X_2, \ldots$ of sets in $\mathcal{S}$ such that

$$X = \bigcup_{k=1}^{\infty} X_k \quad \text{and} \quad \mu(X_k) < \infty \text{ for every } k \in \mathbf{Z}^+.$$

5.19 Example *finite and σ-finite measures*

- Lebesgue measure on the interval $[0, 1]$ is a finite measure.
- Lebesgue measure on $\mathbf{R}$ is not a finite measure but is a σ-finite measure.
- Counting measure on $\mathbf{R}$ is not a σ-finite measure (because the countable union of finite sets is a countable set).

The next result will allow us to define the product of two σ-finite measures.

5.20 *measure of cross section is a measurable function*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

(a) $x \mapsto \nu([E]_x)$ is an $\mathcal{S}$-measurable function on X;

(b) $y \mapsto \mu([E]^y)$ is a $\mathcal{T}$-measurable function on Y.

Proof We will prove (a). If $E \in \mathcal{S} \otimes \mathcal{T}$, then $[E]_x \in \mathcal{T}$ for every $x \in X$ (by 5.6); thus the function $x \mapsto \nu([E]_x)$ is well defined on X.

We first consider the case where ν is a finite measure. Let

$$\mathcal{M} = \{E \in \mathcal{S} \otimes \mathcal{T} : x \mapsto \nu([E]_x) \text{ is an } \mathcal{S}\text{-measurable function on } X\}.$$

We need to prove that $\mathcal{M} = \mathcal{S} \otimes \mathcal{T}$.

If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then $\nu([A \times B]_x) = \nu(B)\chi_A(x)$ for every $x \in X$ (by Example 5.5). Thus the function $x \mapsto \nu([A \times B]_x)$ equals the function $\nu(B)\chi_A$ (as a function on X), which is an $\mathcal{S}$-measurable function on X. Hence $\mathcal{M}$ contains all the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$.

Let $\mathcal{A}$ denote the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Suppose $E \in \mathcal{A}$. Then by 5.13(b), E is a union of disjoint measurable rectangles $E_1, \dots, E_n$. Thus

$$\begin{aligned} \nu([E]_x) &= \nu([E_1 \cup \cdots \cup E_n]_x) \\ &= \nu([E_1]_x \cup \cdots \cup [E_n]_x) \\ &= \nu([E_1]_x) + \cdots + \nu([E_n]_x), \end{aligned}$$

where the last equality holds because ν is a measure and $[E_1]_x, \dots, [E_n]_x$ are disjoint. The equation above, when combined with the conclusion of the previous paragraph, shows that $x \mapsto \nu([E]_x)$ is a finite sum of $\mathcal{S}$-measurable functions and thus is an $\mathcal{S}$-measurable function. Hence $E \in \mathcal{M}$. We have now shown that $\mathcal{A} \subset \mathcal{M}$.

Our next goal is to show that $\mathcal{M}$ is a monotone class on $X \times Y$. To do this, first suppose $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{M}$. Then

$$\begin{aligned} \nu\Bigl([\bigcup_{k=1}^{\infty} E_k]_x\Bigr) &= \nu\Bigl(\bigcup_{k=1}^{\infty}([E_k]_x)\Bigr) \\ &= \lim_{k\to\infty} \nu([E_k]_x), \end{aligned}$$

where we have used 2.59. Because the pointwise limit of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable (by 2.48), the equation above shows that $x \mapsto \nu\bigl([\bigcup_{k=1}^{\infty} E_k]_x\bigr)$ is an $\mathcal{S}$-measurable function. Hence $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$. We have now shown that $\mathcal{M}$ is closed under countable increasing unions.

Now suppose $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{M}$. Then

$$\begin{aligned}\nu\Big([\bigcap_{k=1}^{\infty} E_k]_x\Big) &= \nu\Big(\bigcap_{k=1}^{\infty}([E_k]_x)\Big)\\ &= \lim_{k\to\infty} \nu([E_k]_x),\end{aligned}$$

where we have used 2.60 (this is where we use the assumption that ν is a finite measure). Because the pointwise limit of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable (by 2.48), the equation above shows that $x \mapsto \nu\big([\bigcap_{k=1}^{\infty} E_k]_x\big)$ is an $\mathcal{S}$-measurable function. Hence $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$. We have now shown that $\mathcal{M}$ is closed under countable decreasing intersections.

We have shown that $\mathcal{M}$ is a monotone class that contains the algebra $\mathcal{A}$ of all finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ [by 5.13(a), $\mathcal{A}$ is indeed an algebra]. The Monotone Class Theorem (5.17) implies that $\mathcal{M}$ contains the smallest σ-algebra containing $\mathcal{A}$. In other words, $\mathcal{M}$ contains $\mathcal{S} \otimes \mathcal{T}$. This conclusion completes the proof of (a) in the case where ν is a finite measure.

Now consider the case where ν is a σ-finite measure. Thus there exists a sequence $Y_1, Y_2, \ldots$ of sets in $\mathcal{T}$ such that $\bigcup_{k=1}^{\infty} Y_k = Y$ and $\nu(Y_k) < \infty$ for each $k \in \mathbf{Z}^+$. Replacing each Y_k by $Y_1 \cup \cdots \cup Y_k$, we can assume that $Y_1 \subset Y_2 \subset \cdots$. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

$$\nu([E]_x) = \lim_{k\to\infty} \nu([E \cap (X \times Y_k)]_x).$$

The function $x \mapsto \nu\big([E \cap (X \times Y_k)]_x\big)$ is an $\mathcal{S}$-measurable function on X, as follows by considering the finite measure obtained by restricting ν to the σ-algebra on Y_k consisting of sets in $\mathcal{T}$ that are contained in Y_k. The equation above now implies that $x \mapsto \nu([E]_x)$ is an $\mathcal{S}$-measurable function on X, completing the proof of (a).

The proof of (b) is similar. ∎

5.21 Definition ***integration notation***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $g \colon X \to [-\infty, \infty]$ is a function. The notation

$$\int g(x)\, d\mu(x) \quad \text{means} \quad \int g\, d\mu,$$

where $d\mu(x)$ indicates that variables other than x should be treated as constants.

5.22 Example *integrals*

If λ is Lebesgue measure on $[0,4]$, then

$$\int_{[0,4]} (x^2 + y)\, d\lambda(y) = 4x^2 + 8 \quad \text{and} \quad \int_{[0,4]} (x^2 + y)\, d\lambda(x) = \frac{64}{3} + 4y.$$

The intent in the next definition is that $\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x)$ is defined only when the inner integral and then the outer integral both make sense.

5.23 Definition ***iterated integrals***

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are measure spaces and $f\colon X \times Y \to \mathbf{R}$ is a function. Then

$$\int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x) \quad \text{means} \quad \int_X \Big(\int_Y f(x, y)\, d\nu(y) \Big)\, d\mu(x).$$

In other words, to compute $\int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x)$, first (temporarily) fix $x \in X$ and compute $\int_Y f(x, y)\, d\nu(y)$ [if this integral makes sense]. Then compute the integral with respect to μ of the function $x \mapsto \int_Y f(x, y)\, d\nu(y)$ [if this integral makes sense].

5.24 Example *iterated integrals*

If λ is Lebesgue measure on $[0, 4]$, then

$$\begin{aligned} \int_{[0,4]} \int_{[0,4]} (x^2 + y)\, d\lambda(y)\, d\lambda(x) &= \int_{[0,4]} (4x^2 + 8)\, d\lambda(x) \\ &= \frac{352}{3} \end{aligned}$$

and

$$\begin{aligned} \int_{[0,4]} \int_{[0,4]} (x^2 + y)\, d\lambda(x)\, d\lambda(y) &= \int_{[0,4]} \Big(\frac{64}{3} + 4y \Big)\, d\lambda(y) \\ &= \frac{352}{3}. \end{aligned}$$

The two iterated integrals in this example turned out to both equal $\frac{352}{3}$, even though they do not look alike in the intermediate step of the evaluation. As we will see in the next section, this equality of integrals when changing the order of integration is not a coincidence.

The definition of $(\mu \times \nu)(E)$ given below makes sense because the inner integral below equals $\nu([E]_x)$, which makes sense by 5.6 (or use 5.9), and then the outer integral makes sense by 5.20(a).

The restriction in the definition below to σ-finite measures is not bothersome because the main results we seek are not valid without this hypothesis (see Example 5.30 in the next section).

5.25 Definition ***product of two measures;*** $\mu \times \nu$

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. For $E \in \mathcal{S} \otimes \mathcal{T}$, define $(\mu \times \nu)(E)$ by

$$(\mu \times \nu)(E) = \int_X \int_Y \chi_E(x, y)\, d\nu(y)\, d\mu(x).$$

5.26 Example *measure of a rectangle*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then

$$\begin{aligned}(\mu \times \nu)(A \times B) &= \int_X \int_Y \chi_{A \times B}(x, y)\, d\nu(y)\, d\mu(x) \\ &= \int_X \nu(B)\chi_A(x)\, d\mu(x) \\ &= \mu(A)\nu(B).\end{aligned}$$

Thus product measure of a measurable rectangle is the product of the measures of the corresponding sets.

For $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ σ-finite measure spaces, we defined the product $\mu \times \nu$ to be a function from $\mathcal{S} \otimes \mathcal{T}$ to $[0, \infty]$ (see 5.25). Now we show that this function is a measure.

5.27 ***product of two measures is a measure***

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Then $\mu \times \nu$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$.

Proof Clearly $(\mu \times \nu)(\varnothing) = 0$.

Suppose $E_1, E_2, \ldots$ is a disjoint sequence of sets in $\mathcal{S} \otimes \mathcal{T}$. Then

$$\begin{aligned}(\mu \times \nu)\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) &= \int_X \nu\Bigl(\bigl[\bigcup_{k=1}^{\infty} E_k\bigr]_x\Bigr)\, d\mu(x) \\ &= \int_X \nu\Bigl(\bigcup_{k=1}^{\infty} \bigl([E_k]_x\bigr)\Bigr)\, d\mu(x) \\ &= \int_X \Bigl(\sum_{k=1}^{\infty} \nu([E_k]_x)\Bigr)\, d\mu(x) \\ &= \sum_{k=1}^{\infty} \int_X \nu([E_k]_x)\, d\mu(x) \\ &= \sum_{k=1}^{\infty} (\mu \times \nu)(E_k),\end{aligned}$$

where the fourth equality follows from the Monotone Convergence Theorem (3.11; or see Exercise 10 in Section 3A). The equation above shows that $\mu \times \nu$ satisfies the countable additivity condition required for a measure. ■

EXERCISES 5A

1 Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Prove that if A is a nonempty subset of X and B is a nonempty subset of Y such that $A \times B \in \mathcal{S} \otimes \mathcal{T}$, then $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

2 Suppose $(X, \mathcal{S})$ is a measurable space. Prove that if $E \in \mathcal{S} \otimes \mathcal{S}$, then

$$\{x \in X : (x, x) \in E\} \in \mathcal{S}.$$

3 Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbf{R}$. Show that there exists a set $E \subset \mathbf{R} \times \mathbf{R}$ such that $[E]_a \in \mathcal{B}$ and $[E]^a \in \mathcal{B}$ for every $a \in \mathbf{R}$, but $E \notin \mathcal{B} \otimes \mathcal{B}$.

4 Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Prove that if $f \colon X \to \mathbf{R}$ is $\mathcal{S}$-measurable and $g \colon Y \to \mathbf{R}$ is $\mathcal{T}$-measurable and $h \colon X \times Y \to \mathbf{R}$ is defined by $h(x, y) = f(x)g(y)$, then h is $(\mathcal{S} \otimes \mathcal{T})$-measurable.

5 Verify the assertion in Example 5.11 that the collection of finite unions of intervals of $\mathbf{R}$ is closed under complementation.

6 Verify the assertion in Example 5.12 that the collection of countable unions of intervals of $\mathbf{R}$ is not closed under complementation.

7 Suppose $\mathcal{A}$ is a nonempty collection of subsets of a set W. Show that $\mathcal{A}$ is an algebra on W if and only if $\mathcal{A}$ is closed under finite intersections and under complementation.

8 Suppose μ is a measure on a measurable space $(X, \mathcal{S})$. Prove that the following are equivalent:

(a) The measure μ is σ-finite.

(b) There exists an increasing sequence $X_1 \subset X_2 \subset \cdots$ of sets in $\mathcal{S}$ such that $X = \bigcup_{k=1}^{\infty} X_k$ and $\mu(X_k) < \infty$ for every $k \in \mathbf{Z}^+$.

(c) There exists a disjoint sequence $X_1, X_2, X_3, \ldots$ of sets in $\mathcal{S}$ such that $X = \bigcup_{k=1}^{\infty} X_k$ and $\mu(X_k) < \infty$ for every $k \in \mathbf{Z}^+$.

9 Suppose μ and ν are σ-finite measures. Prove that $\mu \times \nu$ is a σ-finite measure.

10 Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Prove that if ω is a measure on $\mathcal{S} \otimes \mathcal{T}$ such that $\omega(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{S}$ and all $B \in \mathcal{T}$, then $\omega = \mu \times \nu$.
[*The exercise above means that $\mu \times \nu$ is the unique measure on $\mathcal{S} \otimes \mathcal{T}$ that behaves as we expect on measurable rectangles.*]

5B Iterated Integrals

Tonelli's Theorem

Relook at Example 5.24 in the previous section and notice that the value of the iterated integral was unchanged when we switched the order of integration, even though switching the order of integration led to different intermediate results. Our next result states that the order of integration can be switched if the function being integrated is nonnegative and the measures are σ-finite.

5.28 *Tonelli's Theorem*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f \colon X \times Y \to [0, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable. Then

(a) $x \mapsto \int_Y f(x, y)\, d\nu(y)$ is an $\mathcal{S}$-measurable function on X,

(b) $y \mapsto \int_X f(x, y)\, d\mu(x)$ is a $\mathcal{T}$-measurable function on Y,

and

$$\int_{X \times Y} f\, d(\mu \times \nu) = \int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x, y)\, d\mu(x)\, d\nu(y).$$

Proof We begin by considering the special case where $f = \chi_E$ for some $E \in \mathcal{S} \otimes \mathcal{T}$. In this case,

$$\int_Y \chi_E(x, y)\, d\nu(y) = \nu([E]_x) \text{ for every } x \in X$$

and

$$\int_X \chi_E(x, y)\, d\mu(x) = \mu([E]^y) \text{ for every } y \in Y.$$

Thus (a) and (b) hold in this case by 5.20.

First assume that μ and ν are finite measures. Let

$$\mathcal{M} = \Big\{E \in \mathcal{S} \otimes \mathcal{T} \colon \int_X \int_Y \chi_E(x, y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X \chi_E(x, y)\, d\mu(x)\, d\nu(y)\Big\}.$$

If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then $A \times B \in \mathcal{M}$ because both sides of the equation defining $\mathcal{M}$ equal $\mu(A)\nu(B)$.

Let $\mathcal{A}$ denote the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Then 5.13(b) implies that every element of $\mathcal{A}$ is a disjoint union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. The previous paragraph now implies $\mathcal{A} \subset \mathcal{M}$.

The Monotone Convergence Theorem (3.11) implies that $\mathcal{M}$ is closed under countable increasing unions. The Bounded Convergence Theorem (3.26) implies that $\mathcal{M}$ is closed under countable decreasing intersections (this is where we use the assumption that μ and ν are finite measures).

We have shown that $\mathcal{M}$ is a monotone class that contains the algebra $\mathcal{A}$ of all finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ [by 5.13(a), $\mathcal{A}$ is indeed an algebra].

The Monotone Class Theorem (5.17) implies that $\mathcal{M}$ contains the smallest σ-algebra containing $\mathcal{A}$. In other words, $\mathcal{M}$ contains $\mathcal{S} \otimes \mathcal{T}$. Thus

5.29 $$\int_X \int_Y \chi_E(x,y)\,d\nu(y)\,d\mu(x) = \int_Y \int_X \chi_E(x,y)\,d\mu(x)\,d\nu(y)$$

for every $E \in \mathcal{S} \otimes \mathcal{T}$.

Now relax the assumption that μ and ν are finite measures. Write X as an increasing union of sets $X_1 \subset X_2 \subset \cdots$ in $\mathcal{S}$ with finite measure, and write Y as an increasing union of sets $Y_1 \subset Y_2 \subset \cdots$ in $\mathcal{T}$ with finite measure. Suppose $E \in \mathcal{S} \otimes \mathcal{T}$. Applying the finite-measure case to the situation where the measures and the σ-algebras are restricted to X_j and Y_k, we can conclude that 5.29 holds with E replaced by $E \cap (X_j \times Y_k)$ for all $j, k \in \mathbf{Z}^+$. Fix $k \in \mathbf{Z}^+$ and use the Monotone Convergence Theorem (3.11) to conclude that 5.29 holds with E replaced by $E \cap (X \times Y_k)$ for all $k \in \mathbf{Z}^+$. One more use of the Monotone Convergence Theorem then shows that

$$\int_{X\times Y} \chi_E\,d(\mu\times\nu) = \int_X \int_Y \chi_E(x,y)\,d\nu(y)\,d\mu(x) = \int_Y \int_X \chi_E(x,y)\,d\mu(x)\,d\nu(y)$$

for all $E \in \mathcal{S} \otimes \mathcal{T}$, where the first equality above comes from the definition of $(\mu \times \nu)(E)$ (see 5.25).

Now we turn from characteristic functions to the general case of an $\mathcal{S} \otimes \mathcal{T}$-measurable function $f\colon X \times Y \to [0,\infty]$. Define a sequence $f_1, f_2, \ldots$ of simple $\mathcal{S} \otimes \mathcal{T}$-measurable functions from $X \times Y$ to $[0,\infty)$ by

$$f_k(x,y) = \begin{cases} \dfrac{m}{2^k} & \text{if } f(x,y) < k \text{ and } m \text{ is the integer with } f(x,y) \in \left[\dfrac{m}{2^k}, \dfrac{m+1}{2^k}\right), \\ k & \text{if } f(x,y) \geq k. \end{cases}$$

Note that

$$0 \leq f_1(x,y) \leq f_2(x,y) \leq f_3(x,y) \leq \cdots \quad \text{and} \quad \lim_{k\to\infty} f_k(x,y) = f(x,y)$$

for all $(x,y) \in X \times Y$.

Each f_k is a finite sum of functions of the form $c\chi_E$, where $c \in \mathbf{R}$ and $E \in \mathcal{S} \otimes \mathcal{T}$. Thus the conclusions of this theorem hold for each function f_k.

The Monotone Convergence Theorem implies that

$$\int_Y f(x,y)\,d\nu(y) = \lim_{k\to\infty} \int_Y f_k(x,y)\,d\nu(y)$$

for every $x \in X$. Thus the function $x \mapsto \int_Y f(x,y)\,d\nu(y)$ is the pointwise limit on X of a sequence of $\mathcal{S}$-measurable functions. Hence (a) holds, as does (b) for similar reasons.

The last line in the statement of this theorem holds for each f_k. The Monotone Convergence Theorem now implies that the last line in the statement of this theorem holds for f, completing the proof. ∎

See Exercise 1 in this section for an example (with finite measures) showing that Tonelli's Theorem can fail without the hypothesis that the function being integrated is nonnegative. The next example shows that the hypothesis of σ-finite measures also cannot be eliminated.

5.30 Example *Tonelli's Theorem can fail without the hypothesis of σ-finite*

Suppose $\mathcal{B}$ is the σ-algebra of Borel subsets of $[0,1]$, λ is Lebesgue measure on $([0,1],\mathcal{B})$, and μ is counting measure on $([0,1],\mathcal{B})$. Let D denote the diagonal of $[0,1]\times[0,1]$; in other words,

$$D = \{(x,x) : x \in [0,1]\}.$$

Then

$$\int_{[0,1]}\int_{[0,1]} \chi_D(x,y)\,d\mu(y)\,d\lambda(x) = \int_{[0,1]} 1\,d\lambda = 1,$$

but

$$\int_{[0,1]}\int_{[0,1]} \chi_D(x,y)\,d\lambda(x)\,d\mu(y) = \int_{[0,1]} 0\,d\mu = 0.$$

The following useful corollary of Tonelli's Theorem states that we can switch the order of summation in a double-sum of nonnegative numbers. Exercise 2 asks you to find a double-sum of real numbers in which switching the order of summation changes the value of the double sum.

5.31 ***double sums of nonnegative numbers***

If $\{x_{j,k} : j,k \in \mathbf{Z}^+\}$ is a doubly indexed collection of nonnegative numbers, then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_{j,k} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} x_{j,k}.$$

Proof Apply Tonelli's Theorem (5.28) to $\mu\times\mu$, where μ is counting measure on $\mathbf{Z}^+$. ∎

Fubini's Theorem

Our next goal is Fubini's Theorem, which has the same conclusions as Tonelli's Theorem but has a different hypothesis. Tonelli's Theorem requires the function being integrated to be nonnegative. Fubini's Theorem instead requires the integral of the absolute value of the function to be finite. When using Fubini's Theorem to evaluate the integral of f, you will usually first use Tonelli's Theorem as applied to $|f|$ to verify the hypothesis of Fubini's Theorem.

Historically, Fubini's Theorem (proved in 1907) came before Tonelli's Theorem (proved in 1909). However, presenting Tonelli's Theorem first, as is done here, seems to lead to simpler proofs and better understanding. The hard work here went into proving Tonelli's Theorem; thus our proof of Fubini's Theorem consists mainly of bookkeeping details.

As you will see in the proof of Fubini's Theorem, the function in 5.32(a) is defined only for almost every $x \in X$ and the function in 5.32(b) is defined only for almost every $y \in Y$. For convenience, you can think of these functions as equaling 0 on the sets of measure 0 on which they are otherwise undefined.

5.32 *Fubini's Theorem*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f \colon X \times Y \to [-\infty, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable and $\int_{X \times Y} |f| \, d(\mu \times \nu) < \infty$. Then

$$\int_Y |f(x,y)| \, d\nu(y) < \infty \text{ for almost every } x \in X$$

and

$$\int_X |f(x,y)| \, d\mu(x) < \infty \text{ for almost every } y \in Y.$$

Furthermore,

(a) $$x \mapsto \int_Y f(x,y) \, d\nu(y) \text{ is an } \mathcal{S}\text{-measurable function on } X,$$

(b) $$y \mapsto \int_X f(x,y) \, d\mu(x) \text{ is a } \mathcal{T}\text{-measurable function on } Y,$$

and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x) = \int_Y \int_X f(x,y) \, d\mu(x) \, d\nu(y).$$

Proof Tonelli's Theorem (5.28) applied to the nonnegative function $|f|$ implies that $x \mapsto \int_Y |f(x,y)| \, d\nu(y)$ is an $\mathcal{S}$-measurable function on X. Hence

$$\left\{x \in X : \int_Y |f(x,y)| \, d\nu(y) = \infty\right\} \in \mathcal{S}.$$

Tonelli's Theorem applied to $|f|$ also tells us that

$$\int_X \int_Y |f(x,y)| \, d\nu(y) \, d\mu(x) < \infty$$

because the iterated integral above equals $\int_{X \times Y} |f| \, d(\mu \times \nu)$. The inequality above implies that

$$\mu\left(\left\{x \in X : \int_Y |f(x,y)| \, d\nu(y) = \infty\right\}\right) = 0.$$

Recall that f^+ and f^- are nonnegative $\mathcal{S} \otimes \mathcal{T}$-measurable functions such that $|f| = f^+ + f^-$ and $f = f^+ - f^-$ (see 3.17). Applying Tonelli's Theorem to f^+ and f^-, we see that

5.33 $$x \mapsto \int_Y f^+(x,y) \, d\nu(y) \quad \text{and} \quad x \mapsto \int_Y f^-(x,y) \, d\nu(y)$$

are $\mathcal{S}$-measurable functions from X to $[0,\infty]$. Because $f^+ \le |f|$ and $f^- \le |f|$, the sets $\{x \in X : \int_Y f^+(x,y)\,d\nu(y) = \infty\}$ and $\{x \in X : \int_Y f^-(x,y)\,d\nu(y) = \infty\}$ have μ-measure 0. Thus the intersection of these two sets, which is the set of $x \in X$ such that $\int_Y f(x,y)\,d\nu(y)$ is not defined, also has μ-measure 0.

Subtracting the second function in 5.33 from the first function in 5.33, we see that the function that we define to be 0 for those $x \in X$ where we encounter $\infty - \infty$ (a set of μ-measure 0, as noted above) and that equals $\int_Y f(x,y)\,d\nu(y)$ elsewhere is an $\mathcal{S}$-measurable function on X.

Now

$$\begin{aligned}\int_{X\times Y} f\,d(\mu\times\nu) &= \int_{X\times Y} f^+\,d(\mu\times\nu) - \int_{X\times Y} f^-\,d(\mu\times\nu)\\ &= \int_X\int_Y f^+(x,y)\,d\nu(y)\,d\mu(x) - \int_X\int_Y f^-(x,y)\,d\nu(y)\,d\mu(x)\\ &= \int_X\int_Y \big(f^+(x,y) - f^-(x,y)\big)\,d\nu(y)\,d\mu(x)\\ &= \int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x),\end{aligned}$$

where the first line above comes from the definition of the integral of a function that is not nonnegative (note that neither of the two terms on the right side of the first line equals ∞ because $\int_{X\times Y}|f|\,d(\mu\times\nu) < \infty$) and the second line comes from applying Tonelli's Theorem to f^+ and f^-.

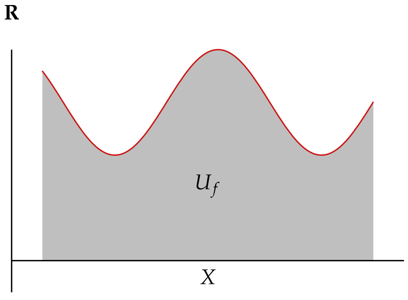
We have now proved all aspects of Fubini's Theorem that involve integrating first over Y. The same procedure provides proofs for the aspects of Fubini's theorem that involve integrating first over X. ∎

Area Under Graph

5.34 Definition ***region under the graph***

Suppose X is a set and $f: X \to [0,\infty]$ is a function. Then the *region under the graph* of f, denoted U_f, is defined by

$$U_f = \{(x,t) \in X \times (0,\infty) : 0 < t < f(x)\}.$$



The figure indicates why we call U_f the region under the graph of f, even in cases when X is not a subset of $\mathbf{R}$. Similarly, the informal term *area* in the next paragraph should remind you of the area in the figure, even though we are really dealing with the measure of U_f in a product space.

The first equality in the result below can be thought of as recovering Riemann's conception of the integral as the area under the graph (although now in a much more general context with arbitrary σ-finite measures). The second equality in the result below can be thought of as reinforcing Lebesgue's conception of computing the area under a curve by integrating in the direction perpendicular to Riemann's.

5.35 *area under the graph of a function equals the integral*

Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $f\colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $(0, \infty)$, and let λ denote Lebesgue measure on $\big((0, \infty), \mathcal{B}\big)$. Then $U_f \in \mathcal{S} \otimes \mathcal{B}$ and

$$(\mu \times \lambda)(U_f) = \int_X f \, d\mu = \int_{(0,\infty)} \mu(\{x \in X : t < f(x)\}) \, d\lambda(t).$$

Proof For $k \in \mathbf{Z}^+$, let

$$E_k = \bigcup_{m=0}^{k^2-1} \Big(f^{-1}\big([\tfrac{m}{k}, \tfrac{m+1}{k})\big) \times \big(0, \tfrac{m}{k}\big) \Big) \quad \text{and} \quad F_k = f^{-1}([k, \infty]) \times (0, k).$$

Then E_k is a finite union of $\mathcal{S} \otimes \mathcal{B}$-measurable rectangles and F_k is an $\mathcal{S} \otimes \mathcal{B}$-measurable rectangle. Because

$$U_f = \bigcup_{k=1}^{\infty} (E_k \cup F_k),$$

we conclude that $U_f \in \mathcal{S} \otimes \mathcal{B}$.

Now the definition of the product measure $\mu \times \lambda$ implies that

$$\begin{aligned}(\mu \times \lambda)(U_f) &= \int_X \int_{(0,\infty)} \chi_{U_f}(x, t) \, d\lambda(t) \, d\mu(x) \\ &= \int_X f(x) \, d\mu(x),\end{aligned}$$

which completes the proof of the first equality in the conclusion of this theorem.

Tonelli's Theorem (5.28) tells us that we can interchange the order of integration in the double integral above, getting

$$\begin{aligned}(\mu \times \lambda)(U_f) &= \int_{(0,\infty)} \int_X \chi_{U_f}(x, t) \, d\mu(x) \, d\lambda(t) \\ &= \int_{(0,\infty)} \mu(\{x \in X : t < f(x)\}) \, d\lambda(t),\end{aligned}$$

which completes the proof of the second equality in the conclusion of this theorem. ■

Markov's inequality (4.1) implies that if f and μ are as in the result above, then

$$\mu(\{x \in X : f(x) > t\}) \leq \frac{\int_X f \, d\mu}{t}$$

for all $t > 0$. Thus if $\int_X f \, d\mu < \infty$, then the result above should be considered to be somewhat stronger than Markov's inequality (because $\int_{(0,\infty)} \frac{1}{t} \, d\lambda(t) = \infty$).

EXERCISES 5B

1 (a) Let λ denote Lebesgue measure on $[0,1]$. Show that

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2}\, d\lambda(y)\, d\lambda(x) = \frac{\pi}{4}$$

and

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2}\, d\lambda(x)\, d\lambda(y) = -\frac{\pi}{4}.$$

(b) Explain why (a) violates neither Tonelli's Theorem nor Fubini's Theorem.

2 (a) Give an example of a doubly indexed collection $\{x_{m,n} : m, n \in \mathbf{Z}^+\}$ of real numbers such that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{m,n} = \infty.$$

(b) Explain why (a) violates neither Tonelli's Theorem nor Fubini's Theorem.

3 Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to [0, \infty]$ is a function. Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $(0, \infty)$. Prove that $U_f \in \mathcal{S} \otimes \mathcal{B}$ if and only if f is an $\mathcal{S}$-measurable function.

4 Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to \mathbf{R}$ is a function. Let $\operatorname{graph}(f) \subset X \times \mathbf{R}$ denote the graph of f:

$$\operatorname{graph}(f) = \{\big(x, f(x)\big) : x \in X\}.$$

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbf{R}$. Prove that $\operatorname{graph}(f) \in \mathcal{S} \otimes \mathcal{B}$ if and only if f is an $\mathcal{S}$-measurable function.

5C Lebesgue Integration on $\mathbf{R}^n$

Throughout this section, assume that m and n are positive integers. Thus, for example, 5.36 should include the hypothesis that m and n are positive integers, but theorems and definitions become easier to state without explicitly repeating this hypothesis.

Borel Subsets of $\mathbf{R}^n$

We begin with a quick review of notation and key concepts concerning $\mathbf{R}^n$.

Recall that $\mathbf{R}^n$ is the set of all n-tuples of real numbers:

$$\mathbf{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbf{R}\}.$$

The function $\|\cdot\|_\infty$ from $\mathbf{R}^n$ to $[0, \infty)$ is defined by

$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

For $x \in \mathbf{R}^n$ and $\delta > 0$, the *open cube* $B(x, \delta)$ with side length 2δ is defined by

$$B(x, \delta) = \{y \in \mathbf{R}^n : \|y - x\|_\infty < \delta\}.$$

If $n = 1$, then an open cube is simply a bounded open interval. If $n = 2$, then an open cube might more appropriately be called an open square. However, using the cube terminology for all dimensions has the advantage of not requiring a different word for different dimensions.

A subset G of $\mathbf{R}^n$ is called *open* if for every $x \in G$, there exists $\delta > 0$ such that $B(x, \delta) \subset G$. Equivalently, a subset G of $\mathbf{R}^n$ is called *open* if every element of G is contained in an open cube that is contained in G.

The union of every collection (finite or infinite) of open subsets of $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$. Also, the intersection of every finite collection of open subsets of $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$.

A subset of $\mathbf{R}^n$ is called *closed* if its complement in $\mathbf{R}^n$ is open. A set $A \subset \mathbf{R}^n$ is called *bounded* if $\sup\{\|a\|_\infty : a \in A\} < \infty$.

We adopt the following common convention:

$$\mathbf{R}^m \times \mathbf{R}^n \text{ is identified with } \mathbf{R}^{m+n}.$$

To understand the necessity of this convention, note that $\mathbf{R}^2 \times \mathbf{R} \neq \mathbf{R}^3$ because $\mathbf{R}^2 \times \mathbf{R}$ and $\mathbf{R}^3$ contain different kinds of objects. Specifically, an element of $\mathbf{R}^2 \times \mathbf{R}$ is an ordered pair, the first of which is an element of $\mathbf{R}^2$ and the second of which is an element of $\mathbf{R}$; thus an element of $\mathbf{R}^2 \times \mathbf{R}$ looks like $\big((x_1, x_2), x_3\big)$. An element of $\mathbf{R}^3$ is an ordered triple of real numbers that looks like (x_1, x_2, x_3). However, we can identify $\big((x_1, x_2), x_3\big)$ with (x_1, x_2, x_3) in the obvious way. Thus we say that $\mathbf{R}^2 \times \mathbf{R}$ "equals" $\mathbf{R}^3$. More generally, we make the natural identification of $\mathbf{R}^m \times \mathbf{R}^n$ with $\mathbf{R}^{m+n}$.

To check that you understand the identification discussed above, make sure that you see why $B(x, \delta) \times B(y, \delta) = B\big((x, y), \delta\big)$ for all $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, and $\delta > 0$.

We can now prove that the product of two open sets is an open set.

5.36 *product of open sets is open*

Suppose G_1 is an open subset of $\mathbf{R}^m$ and G_2 is an open subset of $\mathbf{R}^n$. Then $G_1 \times G_2$ is an open subset of $\mathbf{R}^{m+n}$.

Proof Suppose $(x, y) \in G_1 \times G_2$. Then there exists an open cube D in $\mathbf{R}^m$ centered at x and an open cube E in $\mathbf{R}^n$ centered at y such that $D \subset G_1$ and $E \subset G_2$. By reducing the size of either D or E, we can assume that the cubes D and E have the same side length. Thus $D \times E$ is an open cube in $\mathbf{R}^{m+n}$ centered at (x, y) that is contained in $G_1 \times G_2$.

We have shown that an arbitrary point in $G_1 \times G_2$ is the center of an open cube contained in $G_1 \times G_2$. Hence $G_1 \times G_2$ is an open subset of $\mathbf{R}^{m+n}$. ■

When $n = 1$, the definition below of a Borel subset of $\mathbf{R}^1$ agrees with our previous definition (2.29) of a Borel subset of $\mathbf{R}$.

5.37 Definition *Borel set; $\mathcal{B}_n$*

- A *Borel subset* of $\mathbf{R}^n$ is an element of the smallest σ-algebra on $\mathbf{R}^n$ containing all open subsets of $\mathbf{R}^n$.
- The σ-algebra of Borel subsets of $\mathbf{R}^n$ is denoted by $\mathcal{B}_n$.

Recall that a subset of $\mathbf{R}$ is open if and only if it is a countable disjoint union of open intervals. Part (a) in the result below provides a similar result in $\mathbf{R}^n$, although we must give up the disjoint aspect.

5.38 *open sets are countable unions of open cubes*

(a) A subset of $\mathbf{R}^n$ is open in $\mathbf{R}^n$ if and only if it is a countable union of open cubes in $\mathbf{R}^n$.

(b) $\mathcal{B}_n$ is the smallest σ-algebra on $\mathbf{R}^n$ containing all the open cubes in $\mathbf{R}^n$.

Proof We will prove (a), which clearly implies (b).

The proof that a countable union of open cubes is open is left as an exercise for the reader (actually, arbitrary unions of open cubes are open).

To prove the other direction, suppose G is an open subset of $\mathbf{R}^n$. For each $x \in G$, there is an open cube centered at x that is contained in G. Thus there is a smaller cube C_x such that $x \in C_x \subset G$ and all coordinates of the center of C_x are rational numbers and the side length of C_x is a rational number. Now

$$G = \bigcup_{x \in G} C_x.$$

However, there are only countably many distinct cubes whose center has all rational coordinates and whose side length is rational. Thus G is the countable union of open cubes. ■

The next result tells us that the collection of Borel sets from various dimensions fit together nicely.

5.39 ***product of the Borel subsets of $\mathbf{R}^m$ and the Borel subsets of $\mathbf{R}^n$***

$\mathcal{B}_m \otimes \mathcal{B}_n = \mathcal{B}_{m+n}$.

Proof Suppose E is an open cube in $\mathbf{R}^{m+n}$. Thus E is the product of an open cube in $\mathbf{R}^m$ and an open cube in $\mathbf{R}^n$. Hence $E \in \mathcal{B}_m \otimes \mathcal{B}_n$. Thus the smallest σ-algebra containing all the open cubes in $\mathbf{R}^{m+n}$ is contained in $\mathcal{B}_m \otimes \mathcal{B}_n$. Now 5.38(b) implies that $\mathcal{B}_{m+n} \subset \mathcal{B}_m \otimes \mathcal{B}_n$.

To prove the set inclusion in the other direction, temporarily fix an open set G in $\mathbf{R}^n$. Let

$$\mathcal{E} = \{A \subset \mathbf{R}^m : A \times G \in \mathcal{B}_{m+n}\}.$$

Then $\mathcal{E}$ contains every open subset of $\mathbf{R}^m$ (as follows from 5.36). Also, $\mathcal{E}$ is closed under countable unions because

$$\Big(\bigcup_{k=1}^{\infty} A_k\Big) \times G = \bigcup_{k=1}^{\infty} (A_k \times G).$$

Furthermore, $\mathcal{E}$ is closed under complementation because

$$\big(\mathbf{R}^m \setminus A\big) \times G = \Big(\big(\mathbf{R}^m \times \mathbf{R}^n\big) \setminus \big(A \times G\big)\Big) \cap \big(\mathbf{R}^m \times G\big).$$

Thus $\mathcal{E}$ is a σ-algebra on $\mathbf{R}^m$ that contains all open subsets of $\mathbf{R}^m$, which implies that $\mathcal{B}_m \subset \mathcal{E}$. In other words, we have proved that if $A \in \mathcal{B}_m$ and G is an open subset of $\mathbf{R}^n$, then $A \times G \in \mathcal{B}_{m+n}$.

Now temporarily fix a Borel subset A of $\mathbf{R}^m$. Let

$$\mathcal{F} = \{B \subset \mathbf{R}^n : A \times B \in \mathcal{B}_{m+n}\}.$$

The conclusion of the previous paragraph shows that $\mathcal{F}$ contains every open subset of $\mathbf{R}^n$. As in the previous paragraph, we also see that $\mathcal{F}$ is a σ-algebra. Hence $\mathcal{B}_n \subset \mathcal{F}$. In other words, we have proved that if $A \in \mathcal{B}_m$ and $B \in \mathcal{B}_n$, then $A \times B \in \mathcal{B}_{m+n}$. Thus $\mathcal{B}_m \otimes \mathcal{B}_n \subset \mathcal{B}_{m+n}$, completing the proof. ■

The previous result implies a nice associative property. Specifically, if m, n, and p are positive integers, then two applications of 5.39 give

$$(\mathcal{B}_m \otimes \mathcal{B}_n) \otimes \mathcal{B}_p = \mathcal{B}_{m+n} \otimes \mathcal{B}_p = \mathcal{B}_{m+n+p}.$$

Similarly, two more applications of 5.39 give

$$\mathcal{B}_m \otimes (\mathcal{B}_n \otimes \mathcal{B}_p) = \mathcal{B}_m \otimes \mathcal{B}_{n+p} = \mathcal{B}_{m+n+p}.$$

Thus $(\mathcal{B}_m \otimes \mathcal{B}_n) \otimes \mathcal{B}_p = \mathcal{B}_m \otimes (\mathcal{B}_n \otimes \mathcal{B}_p)$; hence we can dispense with parentheses when taking products of more than two Borel σ-algebras. More generally, we could have defined $\mathcal{B}_m \otimes \mathcal{B}_n \otimes \mathcal{B}_p$ directly as the smallest σ-algebra on $\mathbf{R}^{m+n+p}$ containing $\{A \times B \times C : A \in \mathcal{B}_m, B \in \mathcal{B}_n, C \in \mathcal{B}_p\}$ and obtained the same σ-algebra (see Exercise 3 in this section).

Lebesgue Measure on $\mathbf{R}^n$

5.40 Definition ***Lebesgue measure;*** λ_n

Lebesgue measure on $\mathbf{R}^n$ is denoted by λ_n and is defined inductively by

$$\lambda_n = \lambda_{n-1} \times \lambda_1,$$

where λ_1 is Lebesgue measure on $(\mathbf{R}, \mathcal{B}_1)$.

Because $\mathcal{B}_n = \mathcal{B}_{n-1} \otimes \mathcal{B}_1$ (by 5.39), the measure λ_n is defined on the Borel subsets of $\mathbf{R}^n$. Thinking of a typical point in $\mathbf{R}^n$ as (x, y), where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$, we can use the definition of the product of two measures (5.25) to write

$$\lambda_n(E) = \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}} \chi_E(x, y)\, d\lambda_1(y)\, d\lambda_{n-1}(x)$$

for $E \in \mathcal{B}_n$. Of course, we could use Tonelli's Theorem (5.28) to interchange the order of integration in the equation above.

Because Lebesgue measure is the most commonly used measure, mathematicians often dispense with explicitly displaying the measure and just use a variable name. In other words, if no measure is explicitly displayed in an integral and the context indicates no other measure, then you should assume that the measure involved is Lebesgue measure in the appropriate dimension. For example, the result of interchanging the order of integration in the equation above could be written as

$$\lambda_n(E) = \int_{\mathbf{R}} \int_{\mathbf{R}^{n-1}} \chi_E(x, y)\, dx\, dy$$

for $E \in \mathcal{B}_n$; here dx means $d\lambda_{n-1}(x)$ and dy means $d\lambda_1(y)$.

In the equations above giving formulas for $\lambda_n(E)$, the integral over $\mathbf{R}^{n-1}$ could be rewritten as an iterated integral over $\mathbf{R}^{n-2}$ and $\mathbf{R}$, and that process could be repeated until reaching iterated integrals only over $\mathbf{R}$. Tonelli's Theorem could then be used repeatedly to swap the order of pairs of those integrated integrals, leading to iterated integrals in any order.

Similar comments apply to integrating functions on $\mathbf{R}^n$ other than characteristic functions. For example, if $f \colon \mathbf{R}^3 \to \mathbf{R}$ is a $\mathcal{B}_3$-measurable function such that either $f \geq 0$ or $\int_{\mathbf{R}^3} |f|\, d\lambda_3 < \infty$, then by either Tonelli's Theorem or Fubini's Theorem we have

$$\int_{\mathbf{R}^3} f\, d\lambda_3 = \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x_1, x_2, x_3)\, dx_j\, dx_k\, dx_m,$$

where j, k, m is any permutation of $1, 2, 3$.

Although we defined λ_n to be $\lambda_{n-1} \times \lambda_1$, we could have defined λ_n to be $\lambda_j \times \lambda_k$ for any positive integers j, k with $j + k = n$. This potentially different definition would have led to the same σ-algebra $\mathcal{B}_n$ (by 5.39) and to the same measure λ_n [because both potential definitions of $\lambda_n(E)$ can be written as identical iterations of n integrals with respect to λ_1].

Volume of Unit Ball in $\mathbf{R}^n$

The proof of the next result provides good experience in working with the Lebesgue measure λ_n. Recall that $tE = \{tx : x \in E\}$.

5.41 *measure of a dilation*

Suppose $t > 0$. If $E \in \mathcal{B}_n$, then $tE \in \mathcal{B}_n$ and $\lambda_n(tE) = t^n \lambda_n(E)$.

Proof Let

$$\mathcal{E} = \{E \in \mathcal{B}_n : tE \in \mathcal{B}_n\}.$$

Then $\mathcal{E}$ contains every open subset of $\mathbf{R}^n$ (because if E is open in $\mathbf{R}^n$ then tE is open in $\mathbf{R}^n$). Also, $\mathcal{E}$ is closed under complementation and countable unions because

$$t(\mathbf{R}^n \setminus E) = \mathbf{R}^n \setminus (tE) \quad \text{and} \quad t\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \bigcup_{k=1}^{\infty} (tE_k).$$

Hence $\mathcal{E}$ is a σ-algebra on $\mathbf{R}^n$ containing the open subsets of $\mathbf{R}^n$. Thus $\mathcal{E} = \mathcal{B}_n$. In other words, $tE \in \mathcal{B}_n$ for all $E \in \mathcal{B}_n$.

To prove $\lambda_n(tE) = t^n \lambda_n(E)$, first consider the case $n = 1$. Lebesgue measure on $\mathbf{R}$ is a restriction of outer measure. The outer measure of a set is determined by the sum of the lengths of countable collections of intervals whose union contains the set. Multiplying the set by t corresponds to multiplying each such interval by t, which multiplies the length of each such interval by t. In other words, $\lambda_1(tE) = t\lambda_1(E)$.

Now assume $n > 1$. We will use induction on n and assume that the desired result holds for $n - 1$. If $A \in \mathcal{B}_{n-1}$ and $B \in \mathcal{B}_1$, then

$$\begin{aligned} \lambda_n\big(t(A \times B)\big) &= \lambda_n\big((tA) \times (tB)\big) \\ &= \lambda_{n-1}(tA) \cdot \lambda_1(tB) \\ &= t^{n-1}\lambda_{n-1}(A) \cdot t\lambda_1(B) \\ &= t^n \lambda_n(A \times B), \end{aligned} \tag{5.42}$$

giving the desired result for $A \times B$.

For $m \in \mathbf{Z}^+$, let C_m be the open cube in $\mathbf{R}^n$ centered at the origin and with side length m. Let

$$\mathcal{E}_m = \{E \in \mathcal{B}_n : E \subset C_m \text{ and } \lambda_n(tE) = t^n \lambda_n(E)\}.$$

From 5.42 and using 5.13(b), we see that finite unions of measurable rectangles contained in C_m are in $\mathcal{E}_m$. You should verify that $\mathcal{E}_m$ is closed under countable increasing unions (use 2.59) and countable decreasing intersections (use 2.60, whose finite measure condition holds because we are working inside C_m). From 5.13 and the Monotone Class Theorem (5.17), we conclude that $\mathcal{E}_m$ is the σ-algebra on C_m consisting of Borel subsets of C_m. Thus $\lambda_n(tE) = t^n \lambda_n(E)$ for all $E \in \mathcal{B}_n$ such that $E \subset C_m$.

Now suppose $E \in \mathcal{B}_n$. Then 2.59 implies that

$$\lambda_n(tE) = \lim_{m\to\infty} \lambda_n\big(t(E \cap C_m)\big) = t^n \lim_{m\to\infty} \lambda_n(E \cap C_m) = t^n \lambda_n(E),$$

as desired. ∎

5.43 Definition *open unit ball in* $\mathbf{R}^n$

The *open unit ball* in $\mathbf{R}^n$ is denoted by $\mathbf{B}_n$ and is defined by

$$\mathbf{B}_n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : {x_1}^2 + \cdots + {x_n}^2 < 1\}.$$

The open unit ball $\mathbf{B}_n$ is open in $\mathbf{R}^n$ (as you should verify) and thus is in the collection $\mathcal{B}_n$ of Borel sets.

5.44 *volume of the unit ball in* $\mathbf{R}^n$

$$\lambda_n(\mathbf{B}_n) = \begin{cases} \dfrac{\pi^{n/2}}{(n/2)!} & \text{if } n \text{ is even,} \\[2ex] \dfrac{2^{(n+1)/2}\,\pi^{(n-1)/2}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot n} & \text{if } n \text{ is odd.} \end{cases}$$

Proof Because $\lambda_1(\mathbf{B}_1) = 2$ and $\lambda_2(\mathbf{B}_2) = \pi$, the claimed formula is correct when $n = 1$ and when $n = 2$.

Now assume that $n > 2$. We will use induction on n, assuming that the claimed formula is true for smaller values of n. Think of $\mathbf{R}^n = \mathbf{R}^2 \times \mathbf{R}^{n-2}$ and $\lambda_n = \lambda_2 \times \lambda_{n-2}$. Then

5.45 $$\lambda_n(\mathbf{B}_n) = \int_{\mathbf{R}^2} \int_{\mathbf{R}^{n-2}} \chi_{\mathbf{B}_n}(x, y)\, dy\, dx.$$

Temporarily fix $x = (x_1, x_2) \in \mathbf{R}^2$. If ${x_1}^2 + {x_2}^2 \geq 1$, then $\chi_{\mathbf{B}_n}(x, y) = 0$ for all $y \in \mathbf{R}^{n-2}$. If ${x_1}^2 + {x_2}^2 < 1$ and $y \in \mathbf{R}^{n-2}$, then $\chi_{\mathbf{B}_n}(x, y) = 1$ if and only if $y \in (1 - {x_1}^2 - {x_2}^2)^{1/2}\mathbf{B}_{n-2}$. Thus the inner integral in 5.45 equals

$$\lambda_{n-2}\Big((1 - {x_1}^2 - {x_2}^2)^{1/2}\mathbf{B}_{n-2}\Big)\chi_{\mathbf{B}_2}(x),$$

which by 5.41 equals

$$(1 - {x_1}^2 - {x_2}^2)^{(n-2)/2}\lambda_{n-2}(\mathbf{B}_{n-2})\chi_{\mathbf{B}_2}(x).$$

Thus 5.45 becomes the equation

$$\lambda_n(\mathbf{B}_n) = \lambda_{n-2}(\mathbf{B}_{n-2}) \int_{\mathbf{B}_2} (1 - {x_1}^2 - {x_2}^2)^{(n-2)/2}\, d\lambda_2(x_1, x_2).$$

To evaluate this integral, switch to the usual polar coordinates that you learned about in calculus ($d\lambda_2 = r\, dr\, d\theta$), getting

$$\begin{aligned} \lambda_n(\mathbf{B}_n) &= \lambda_{n-2}(\mathbf{B}_{n-2}) \int_{-\pi}^{\pi} \int_0^1 (1 - r^2)^{(n-2)/2} r\, dr\, d\theta \\ &= \frac{2\pi}{n} \lambda_{n-2}(\mathbf{B}_{n-2}). \end{aligned}$$

The last equation and the induction hypothesis give the desired result. ■

The table here gives the first five values of $\lambda_n(\mathbf{B}_n)$, using 5.44. The last column of this table gives a decimal approximation to $\lambda_n(\mathbf{B}_n)$, accurate to two digits after the decimal point. From this table, you might guess that $\lambda_n(\mathbf{B}_n)$ is an increasing function of n, especially because the smallest cube containing the ball $\mathbf{B}_n$ has n-dimensional Lebesgue measure 2^n. However, Exercise 12 in this section shows that $\lambda_n(\mathbf{B}_n)$ behaves much differently.

n	$\lambda_n(\mathbf{B}_n)$	$\approx \lambda_n(\mathbf{B}_n)$
1	2	2.00
2	π	3.14
3	$4\pi/3$	4.19
4	$\pi^2/2$	4.93
5	$8\pi^2/15$	5.26

Equality of Mixed Partial Derivatives Via Fubini's Theorem

5.46 Definition ***partial derivatives; $D_1 f$ and $D_2 f$***

Suppose G is an open subset of $\mathbf{R}^2$ and $f\colon G \to \mathbf{R}$ is a function. For $(x,y) \in G$, the *partial derivatives* $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ are defined by

$$(D_1 f)(x,y) = \lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t}$$

and

$$(D_2 f)(x,y) = \lim_{t \to 0} \frac{f(x,y+t) - f(x,y)}{t}$$

if these limits exist.

Using the notation for the cross section of a function (see 5.7), we could write the definitions of D_1 and D_2 in the following form:

$$(D_1 f)(x,y) = ([f]^y)'(x) \quad \text{and} \quad (D_2 f)(x,y) = ([f]_x)'(y).$$

5.47 Example *partial derivatives of x^y*

Let $G = \{(x,y) \in \mathbf{R}^2 : x > 0\}$ and define $f\colon G \to \mathbf{R}$ by $f(x,y) = x^y$. Then

$$(D_1 f)(x,y) = yx^{y-1} \quad \text{and} \quad (D_2 f)(x,y) = x^y \ln x,$$

as you should verify. Taking partial derivatives of those partial derivatives, we have

$$\big(D_2(D_1 f)\big)(x,y) = x^{y-1} + yx^{y-1} \ln x$$

and

$$\big(D_1(D_2 f)\big)(x,y) = x^{y-1} + yx^{y-1} \ln x,$$

as you should also verify. The last two equations show that $D_1(D_2 f) = D_2(D_1 f)$ as functions on G.

In the example above, the two mixed partial derivatives turn out to equal each other, even though the intermediate results look quite different. The next result shows that the behavior in the example above is typical rather than a coincidence.

Some proofs of the result below do not use Fubini's Theorem. However, Fubini's Theorem leads to the clean proof below.

The integrals that appear in the proof below make sense because continuous real-valued functions on $\mathbf{R}^2$ are measurable (because for a continuous function, the inverse image of each open set is open) and because continuous real-valued functions on closed bounded subsets of $\mathbf{R}^2$ are bounded.

Although the continuity hypotheses in the result below can be slightly weakened, they cannot be eliminated, as shown by Exercise 14 in this section.

5.48 *equality of mixed partial derivatives*

Suppose G is an open subset of $\mathbf{R}^2$ and $f\colon G \to \mathbf{R}$ is a function such that $D_1 f$, $D_2 f$, $D_1(D_2 f)$, and $D_2(D_1 f)$ all exist and are continuous functions on G. Then

$$D_1(D_2 f) = D_2(D_1 f)$$

on G.

Proof Fix $(a, b) \in G$. For $\delta > 0$, let $S_\delta = [a, a+\delta] \times [b, b+\delta]$. If $S_\delta \subset G$, then

$$\begin{aligned}
\int_{S_\delta} D_1(D_2 f)\, d\lambda_2 &= \int_b^{b+\delta} \int_a^{a+\delta} \big(D_1(D_2 f)\big)(x, y)\, dx\, dy \\
&= \int_b^{b+\delta} \big[(D_2 f)(a+\delta, y) - (D_2 f)(a, y)\big]\, dy \\
&= f(a+\delta, b+\delta) - f(a+\delta, b) - f(a, b+\delta) + f(a, b),
\end{aligned}$$

where the first equality comes from Fubini's Theorem (5.32) and the second and third equalities come from the Fundamental Theorem of Calculus.

A similar calculation of $\int_{S_\delta} D_2(D_1 f)\, d\lambda_2$ yields the same result. Thus

$$\int_{S_\delta} [D_1(D_2 f) - D_2(D_1 f)]\, d\lambda_2 = 0$$

for all δ such that $S_\delta \subset G$. If $\big(D_1(D_2 f)\big)(a, b) > \big(D_2(D_1 f)\big)(a, b)$, then by the continuity of $D_1(D_2 f)$ and $D_2(D_1 f)$, the integrand in the equation above is positive on S_δ for δ sufficiently small, which contradicts the integral above equaling 0. Similarly, the inequality $\big(D_1(D_2 f)\big)(a, b) < \big(D_2(D_1 f)\big)(a, b)$ also contradicts the equation above for small δ. Thus we conclude that

$$\big(D_1(D_2 f)\big)(a, b) = \big(D_2(D_1 f)\big)(a, b),$$

as desired. ∎

EXERCISES 5C

1 Show that a set $G \subset \mathbf{R}^n$ is open in $\mathbf{R}^n$ if and only if for each $(b_1, \ldots, b_n) \in G$, there exists $r > 0$ such that

$$\Big\{(a_1, \ldots, a_n) \in \mathbf{R}^n : \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2} < r\Big\} \subset G.$$

2 Show that there exists a set $E \subset \mathbf{R}^2$ (thinking of $\mathbf{R}^2$ as equal to $\mathbf{R} \times \mathbf{R}$) such that the cross sections $[E]_a$ and $[E]^a$ are open subsets of $\mathbf{R}$ for every $a \in \mathbf{R}$, but $E \notin \mathcal{B}_2$.

3 Suppose $(X, \mathcal{S})$, $(Y, \mathcal{T})$, and $(Z, \mathcal{U})$ are measurable spaces. We can define $\mathcal{S} \otimes \mathcal{T} \otimes \mathcal{U}$ to be the smallest σ-algebra on $X \times Y \times Z$ that contains

$$\{A \times B \times C : A \in \mathcal{S}, B \in \mathcal{T}, C \in \mathcal{U}\}.$$

Prove that if we make the obvious identifications of the products $(X \times Y) \times Z$ and $X \times (Y \times Z)$ with $X \times Y \times Z$, then

$$\mathcal{S} \otimes \mathcal{T} \otimes \mathcal{U} = (\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{U} = \mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{U}).$$

4 Show that Lebesgue measure on $\mathbf{R}^n$ is translation invariant. More precisely, show that if $E \in \mathcal{B}_n$ and $a \in \mathbf{R}^n$, then $a + E \in \mathcal{B}_n$ and $\lambda_n(a + E) = \lambda_n(E)$, where

$$a + E = \{a + x : x \in E\}.$$

5 Suppose $f\colon \mathbf{R}^n \to \mathbf{R}$ is $\mathcal{B}_n$-measurable and $t \in \mathbf{R} \setminus \{0\}$. Define $f_t\colon \mathbf{R}^n \to \mathbf{R}$ by $f_t(x) = f(tx)$.

(a) Prove that f_t is $\mathcal{B}_n$ measurable.

(b) Prove that if $\int_{\mathbf{R}^n} f \, d\lambda_n$ is defined, then

$$\int_{\mathbf{R}^n} f_t \, d\lambda_n = \frac{1}{|t|^n} \int_{\mathbf{R}^n} f \, d\lambda_n.$$

6 Suppose λ denotes Lebesgue measure on $(\mathbf{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$. Show that there exist subsets E and F of $\mathbf{R}^2$ such that

- $F \in \mathcal{L} \otimes \mathcal{L}$ and $(\lambda \times \lambda)(F) = 0$;
- $E \subset F$ but $E \notin \mathcal{L} \otimes \mathcal{L}$.

[*The measure space* $(\mathbf{R}, \mathcal{L}, \lambda)$ *has the property that every subset of a set with measure* 0 *is measurable. This exercise asks you to show that the measure space* $(\mathbf{R}^2, \mathcal{L} \otimes \mathcal{L}, \lambda \times \lambda)$ *does not have this property.*]

7 Suppose $M \in \mathbf{Z}^+$. Verify that the collection of sets $\mathcal{E}_M$ that appears in the proof of 5.41 is a monotone class.

8 Show that the open unit ball in $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$.

9 Suppose G_1 is a nonempty subset of $\mathbf{R}^m$ and G_2 is a nonempty subset of $\mathbf{R}^n$. Prove that $G_1 \times G_2$ is an open subset of $\mathbf{R}^m \times \mathbf{R}^n$ if and only if G_1 is an open subset of $\mathbf{R}^m$ and G_2 is an open subset of $\mathbf{R}^n$.
[*One direction of this result was already proved* (*see* 5.36); *both directions are stated here to make the result look prettier and to be comparable to the next exercise, where neither direction has been proved.*]

10 Suppose F_1 is a nonempty subset of $\mathbf{R}^m$ and F_2 is a nonempty subset of $\mathbf{R}^n$. Prove that $F_1 \times F_2$ is a closed subset of $\mathbf{R}^m \times \mathbf{R}^n$ if and only if F_1 is a closed subset of $\mathbf{R}^m$ and F_2 is a closed subset of $\mathbf{R}^n$.

11 Suppose E is a subset of $\mathbf{R}^m \times \mathbf{R}^n$ and

$$A = \{x \in \mathbf{R}^m : (x, y) \in E \text{ for some } y \in \mathbf{R}^n\}.$$

(a) Prove that if E is an open subset of $\mathbf{R}^m \times \mathbf{R}^n$, then A is an open subset of $\mathbf{R}^m$.

(b) Prove or give a counterexample: If E is a closed subset of $\mathbf{R}^m \times \mathbf{R}^n$, then A is a closed subset of $\mathbf{R}^m$.

12 (a) Prove that $\lim_{n\to\infty} \lambda_n(\mathbf{B}_n) = 0$.

(b) Find the value of n that maximizes $\lambda_n(\mathbf{B}_n)$.

13 For readers familiar with the gamma function Γ: Prove that

$$\lambda_n(\mathbf{B}_n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

for every positive integer n.

14 Define $f\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove that $D_1(D_2 f)$ and $D_2(D_1 f)$ exist everywhere on $\mathbf{R}^2$.

(b) Show that $\big(D_1(D_2 f)\big)(0,0) \neq \big(D_2(D_1 f)\big)(0,0)$.

(c) Explain why (b) does not violate 5.48.

Chapter 6
Banach Spaces

We begin this chapter with a quick review of the essentials of metric spaces. Then we extend our results on measurable functions and integration to complex-valued functions. After that, we rapidly review the framework of vector spaces, which allows us to consider natural collections of measurable functions that are closed under addition and scalar multiplication.

Normed vector spaces and Banach spaces, which are introduced in the third section of this chapter, play a hugely important role in modern analysis. Most interest focuses on linear maps on these vector spaces. Key results about linear maps that we develop in this chapter include the Hahn–Banach Theorem, the Open Mapping Theorem, the Closed Graph Theorem, and the Principle of Uniform Boundedness.



Market square in Lwów, a city that has been in several countries because of changing international boundaries. Before World War I, Lwów was in Austria–Hungary. During the period between World War I and World War II, Lwów was in Poland. During this time, mathematicians in Lwów, particularly Stefan Banach (1892–1945) and his colleagues, developed the basic results of modern functional analysis. After World War II, Lwów was in the USSR. Now Lwów is in Ukraine and is called Lviv.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_6

6A Metric Spaces

Open Sets, Closed Sets, and Continuity

Much of analysis takes place in the context of a metric space, which is a set with a notion of distance that satisfies certain properties. The properties we would like a distance function to have are captured in the next definition, where you should think of $d(f,g)$ as measuring the distance between f and g.

Specifically, we would like the distance between two elements of our metric space to be a nonnegative number that is 0 if and only if the two elements are the same. We would like the distance between two elements not to depend on the order in which we list them. Finally, we would like a triangle inequality (the last bullet point below), which states that the distance between two elements is less than or equal to the sum of the distances obtained when we insert an intermediate element.

Now we are ready for the formal definition.

6.1 Definition ***metric space***

A *metric* on a nonempty set V is a function $d\colon V \times V \to [0,\infty)$ such that

- $d(f,f) = 0$ for all $f \in V$;
- if $f,g \in V$ and $d(f,g) = 0$, then $f = g$;
- $d(f,g) = d(g,f)$ for all $f,g \in V$;
- $d(f,h) \leq d(f,g) + d(g,h)$ for all $f,g,h \in V$.

A *metric space* is a pair (V,d), where V is a nonempty set and d is a metric on V.

6.2 Example *metric spaces*

- Suppose V is a nonempty set. Define d on $V \times V$ by setting $d(f,g)$ to be 1 if $f \neq g$ and to be 0 if $f = g$. Then d is a metric on V.
- Define d on $\mathbf{R} \times \mathbf{R}$ by $d(x,y) = |x - y|$. Then d is a metric on $\mathbf{R}$.
- For $n \in \mathbf{Z}^+$, define d on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$d\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \max\{|x_1 - y_1|,\dots,|x_n - y_n|\}.$$

  Then d is a metric on $\mathbf{R}^n$.
- Define d on $C([0,1]) \times C([0,1])$ by $d(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\}$; here $C([0,1])$ is the set of continuous real-valued functions on $[0,1]$. Then d is a metric on $C([0,1])$.
- Define d on $\ell^1 \times \ell^1$ by $d\big((a_1,a_2,\dots),(b_1,b_2,\dots)\big) = \sum_{k=1}^{\infty} |a_k - b_k|$; here ℓ^1 is the set of sequences $(a_1,a_2,\dots)$ of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$. Then d is a metric on ℓ^1.

The material in this section is probably review for most readers of this book. Thus more details than usual are left to the reader to verify. Verifying those details and doing the exercises is the best way to solidify your understanding of these concepts. You should be able to transfer familiar definitions and proofs from the context of $\mathbf{R}$ or $\mathbf{R}^n$ to the context of a metric space.

This book often uses symbols such as f, g, h as generic elements of a generic metric space because many of the important metric spaces in analysis are sets of functions; for example, see the fourth bullet point of Example 6.2.

We will need to use a metric space's topological features, which we introduce now.

6.3 Definition ***open ball;*** $B(f, r)$

Suppose (V, d) is a metric space, $f \in V$, and $r > 0$.

- The *open ball* centered at f with radius r is denoted $B(f, r)$ and is defined by
$$B(f, r) = \{g \in V : d(f, g) < r\}.$$
- The *closed ball* centered at f with radius r is denoted $\overline{B}(f, r)$ and is defined by
$$\overline{B}(f, r) = \{g \in V : d(f, g) \leq r\}.$$

Abusing terminology, many books (including this one) include phrases such as *suppose V is a metric space* without mentioning the metric d. When that happens, you should assume that a metric d lurks nearby, even if it is not explicitly named.

Our next definition declares a subset of a metric space to be open if every element in the subset is the center of an open ball that is contained in the set.

6.4 Definition ***open set***

A subset G of a metric space V is called *open* if for every $f \in G$, there exists $r > 0$ such that $B(f, r) \subset G$.

6.5 ***open balls are open***

Suppose V is a metric space, $f \in V$, and $r > 0$. Then $B(f, r)$ is an open subset of V.

Proof Suppose $g \in B(f, r)$. We need to show that an open ball centered at g is contained in $B(f, r)$. To do this, note that if $h \in B\bigl(g, r - d(f, g)\bigr)$, then

$$d(f, h) \leq d(f, g) + d(g, h) < d(f, g) + \bigl(r - d(f, g)\bigr) = r,$$

which implies that $h \in B(f, r)$. Thus $B\bigl(g, r - d(f, g)\bigr) \subset B(f, r)$, which implies that $B(f, r)$ is open. ■

Closed sets are defined in terms of open sets.

6.6 Definition ***closed subset***

A subset of a metric space V is called *closed* if its complement in V is open.

For example, each closed ball $\overline{B}(f,r)$ in a metric space is closed, as you are asked to prove in Exercise 3.

Now we define the closure of a subset of a metric space.

6.7 Definition ***closure***

Suppose V is a metric space and $E \subset V$. The *closure* of E, denoted $\overline{E}$, is defined by

$$\overline{E} = \{g \in V : B(g,\varepsilon) \cap E \neq \varnothing \text{ for every } \varepsilon > 0\}.$$

Limits in a metric space are defined by reducing to the context of real numbers, where limits have already been defined.

6.8 Definition ***limit in*** V

Suppose (V,d) is a metric space, $f_1, f_2, \ldots$ is a sequence in V, and $f \in V$. Then

$$\lim_{k\to\infty} f_k = f \text{ means } \lim_{k\to\infty} d(f_k, f) = 0.$$

In other words, a sequence $f_1, f_2, \ldots$ in V converges to $f \in V$ if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that

$$d(f_k, f) < \varepsilon \text{ for all integers } k \geq n.$$

The next result states that the closure of a set is the collection of all limits of elements of the set. Also, a set is closed if and only if it equals its closure. The proof of the next result is left as an exercise that provides good practice in using these concepts.

6.9 ***closure***

Suppose V is a metric space and $E \subset V$. Then

(a) $\overline{E} = \{g \in V : \text{ there exist } f_1, f_2, \ldots \text{ in } E \text{ such that } \lim_{k\to\infty} f_k = g\}$;

(b) $\overline{E}$ is the intersection of all closed subsets of V that contain E;

(c) $\overline{E}$ is a closed subset of V;

(d) E is closed if and only if $\overline{E} = E$;

(e) E is closed if and only if E contains the limit of every convergent sequence of elements of E.

The definition of continuity that follows uses the same pattern as the definition for a function from a subset of $\mathbf{R}$ to $\mathbf{R}$.

6.10 Definition ***continuity***

Suppose (V, d_V) and (W, d_W) are metric spaces and $T\colon V \to W$ is a function.

- For $f \in V$, the function T is called *continuous* at f if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d_W\big(T(f), T(g)\big) < \varepsilon$$
for all $g \in V$ with $d_V(f, g) < \delta$.
- The function T is called *continuous* if T is continuous at f for every $f \in V$.

The next result gives equivalent conditions for continuity. Recall that $T^{-1}(E)$ is called the inverse image of E and is defined to be $\{f \in V : T(f) \in E\}$. Thus the equivalence of the (a) and (c) below could be restated as saying that a function is continuous if and only if the inverse image of every open set is open. The equivalence of the (a) and (d) below could be restated as saying that a function is continuous if and only if the inverse image of every closed set is closed.

6.11 ***equivalent conditions for continuity***

Suppose V and W are metric spaces and $T\colon V \to W$ is a function. Then the following are equivalent:

(a) T is continuous.

(b) $\lim\limits_{k\to\infty} f_k = f$ in V implies $\lim\limits_{k\to\infty} T(f_k) = T(f)$ in W.

(c) $T^{-1}(G)$ is an open subset of V for every open set $G \subset W$.

(d) $T^{-1}(F)$ is a closed subset of V for every closed set $F \subset W$.

Proof We first prove that (b) implies (d). Suppose (b) holds. Suppose F is a closed subset of W. We need to prove that $T^{-1}(F)$ is closed. To do this, suppose $f_1, f_2, \dots$ is a sequence in $T^{-1}(F)$ and $\lim_{k\to\infty} f_k = f$ for some $f \in V$. Because (b) holds, we know that $\lim_{k\to\infty} T(f_k) = T(f)$. Because $f_k \in T^{-1}(F)$ for each $k \in \mathbf{Z}^+$, we know that $T(f_k) \in F$ for each $k \in \mathbf{Z}^+$. Because F is closed, this implies that $T(f) \in F$. Thus $f \in T^{-1}(F)$, which implies that $T^{-1}(F)$ is closed [by 6.9(e)], completing the proof that (b) implies (d).

The proof that (c) and (d) are equivalent follows from the equation
$$T^{-1}(W \setminus E) = V \setminus T^{-1}(E)$$
for every $E \subset W$ and the fact that a set is open if and only if its complement (in the appropriate metric space) is closed.

The proof of the remaining parts of this result are left as an exercise that should help strengthen your understanding of these concepts. ■

Cauchy Sequences and Completeness

The next definition is useful for showing (in some metric spaces) that a sequence has a limit, even when we do not have a good candidate for that limit.

6.12 Definition ***Cauchy sequence***

A sequence $f_1, f_2, \dots$ in a metric space (V, d) is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $d(f_j, f_k) < \varepsilon$ for all integers $j \geq n$ and $k \geq n$.

6.13 ***every convergent sequence is a Cauchy sequence***

Every convergent sequence in a metric space is a Cauchy sequence.

Proof Suppose $\lim_{k\to\infty} f_k = f$ in a metric space (V, d). Suppose $\varepsilon > 0$. Then there exists $n \in \mathbf{Z}^+$ such that $d(f_k, f) < \frac{\varepsilon}{2}$ for all $k \geq n$. If $j, k \in \mathbf{Z}^+$ are such that $j \geq n$ and $k \geq n$, then

$$d(f_j, f_k) \leq d(f_j, f) + d(f, f_k) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Thus $f_1, f_2, \dots$ is a Cauchy sequence, completing the proof. ■

Metric spaces that satisfy the converse of the result above have a special name.

6.14 Definition ***complete metric space***

A metric space V is called *complete* if every Cauchy sequence in V converges to some element of V.

6.15 Example

- All five of the metric spaces in Example 6.2 are complete, as you should verify.
- The metric space $\mathbf{Q}$, with metric defined by $d(x, y) = |x - y|$, is not complete. To see this, for $k \in \mathbf{Z}^+$ let

  $$x_k = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{k!}}.$$

  If $j < k$, then

  $$|x_k - x_j| = \frac{1}{10^{(j+1)!}} + \cdots + \frac{1}{10^{k!}} < \frac{2}{10^{(j+1)!}}.$$

  Thus $x_1, x_2, \dots$ is a Cauchy sequence in $\mathbf{Q}$. However, $x_1, x_2, \dots$ does not converge to an element of $\mathbf{Q}$ because the limit of this sequence would have a decimal expansion $0.110001000000000000000001\dots$ that is neither a terminating decimal nor a repeating decimal. Thus $\mathbf{Q}$ is not a complete metric space.



Entrance to the École Polytechnique (Paris), where Augustin-Louis Cauchy (1789–1857) was a student and a faculty member. Cauchy wrote almost 800 mathematics papers and the highly influential textbook Cours d'Analyse *(published in 1821), which greatly influenced the development of analysis.*

Every nonempty subset of a metric space is a metric space. Specifically, suppose (V, d) is a metric space and U is a nonempty subset of V. Then restricting d to $U \times U$ gives a metric on U. Unless stated otherwise, you should assume that the metric on a subset is this restricted metric that the subset inherits from the bigger set.

Combining the two bullet points in the result below shows that a subset of a complete metric space is complete if and only if it is closed.

6.16 *connection between complete and closed*

(a) A complete subset of a metric space is closed.

(b) A closed subset of a complete metric space is complete.

Proof We begin with a proof of (a). Suppose U is a complete subset of a metric space V. Suppose $f_1, f_2, \ldots$ is a sequence in U that converges to some $g \in V$. Then $f_1, f_2, \ldots$ is a Cauchy sequence in U (by 6.13). Hence by the completeness of U, the sequence $f_1, f_2, \ldots$ converges to some element of U, which must be g (see Exercise 7). Hence $g \in U$. Now 6.9(e) implies that U is a closed subset of V, completing the proof of (a).

To prove (b), suppose U is a closed subset of a complete metric space V. To show that U is complete, suppose $f_1, f_2, \ldots$ is a Cauchy sequence in U. Then $f_1, f_2, \ldots$ is also a Cauchy sequence in V. By the completeness of V, this sequence converges to some $f \in V$. Because U is closed, this implies that $f \in U$ (see 6.9). Thus the Cauchy sequence $f_1, f_2, \ldots$ converges to an element of U, showing that U is complete. Hence (b) has been proved. ∎

EXERCISES 6A

1 Verify that each of the claimed metrics in Example 6.2 is indeed a metric.

2 Prove that every finite subset of a metric space is closed.

3 Prove that every closed ball in a metric space is closed.

4 Suppose V is a metric space.

(a) Prove that the union of each collection of open subsets of V is an open subset of V.

(b) Prove that the intersection of each finite collection of open subsets of V is an open subset of V.

5 Suppose V is a metric space.

(a) Prove that the intersection of each collection of closed subsets of V is a closed subset of V.

(b) Prove that the union of each finite collection of closed subsets of V is a closed subset of V.

6 (a) Prove that if V is a metric space, $f \in V$, and $r > 0$, then $\overline{B(f,r)} \subset \overline{B}(f,r)$.

(b) Give an example of a metric space V, $f \in V$, and $r > 0$ such that $\overline{B(f,r)} \neq \overline{B}(f,r)$.

7 Show that a sequence in a metric space has at most one limit.

8 Prove 6.9.

9 Prove that each open subset of a metric space V is the union of some sequence of closed subsets of V.

10 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $\overline{U} \cup \overline{W} = \overline{U \cup W}$.

11 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $\overline{U} \cap \overline{W} = \overline{U \cap W}$.

12 Suppose (U, d_U), (V, d_V), and (W, d_W) are metric spaces. Suppose also that $T\colon U \to V$ and $S\colon V \to W$ are continuous functions.

(a) Using the definition of continuity, show that $S \circ T\colon U \to W$ is continuous.

(b) Using the equivalence of 6.11(a) and 6.11(b), show that $S \circ T\colon U \to W$ is continuous.

(c) Using the equivalence of 6.11(a) and 6.11(c), show that $S \circ T\colon U \to W$ is continuous.

13 Prove the parts of 6.11 that were not proved in the text.

14 Suppose a Cauchy sequence in a metric space has a convergent subsequence. Prove that the Cauchy sequence converges.

15 Verify that all five of the metric spaces in Example 6.2 are complete metric spaces.

16 Suppose (U, d) is a metric space. Let W denote the set of all Cauchy sequences of elements of U.

(a) For $(f_1, f_2, \ldots)$ and $(g_1, g_2, \ldots)$ in W, define $(f_1, f_2, \ldots) \equiv (g_1, g_2, \ldots)$ to mean that

$$\lim_{k\to\infty} d(f_k, g_k) = 0.$$

Show that $\equiv$ is an equivalence relation on W.

(b) Let V denote the set of equivalence classes of elements of W under the equivalence relation above. For $(f_1, f_2, \ldots) \in W$, let $(f_1, f_2, \ldots)\hat{}$ denote the equivalence class of $(f_1, f_2, \ldots)$. Define $d_V \colon V \times V \to [0, \infty)$ by

$$d_V\big((f_1, f_2, \ldots)\hat{}, (g_1, g_2, \ldots)\hat{}\big) = \lim_{k\to\infty} d(f_k, g_k).$$

Show that this definition of d_V makes sense and that d_V is a metric on V.

(c) Show that (V, d_V) is a complete metric space.

(d) Show that the map from U to V that takes $f \in U$ to $(f, f, f, \ldots)\hat{}$ preserves distances, meaning that

$$d(f, g) = d_V\big((f, f, f, \ldots)\hat{}, (g, g, g, \ldots)\hat{}\big)$$

for all $f, g \in U$.

(e) Explain why (d) shows that every metric space is a subset of some complete metric space.

6B Vector Spaces

Integration of Complex-Valued Functions

Complex numbers were invented so that we can take square roots of negative numbers. The idea is to assume we have a square root of -1, denoted i, that obeys the usual rules of arithmetic. Here are the formal definitions:

6.17 Definition ***complex numbers;*** $\mathbf{C}$

- A *complex number* is an ordered pair (a, b), where $a, b \in \mathbf{R}$, but we write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbf{C}$:
$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$
- *Addition* and *multiplication* on $\mathbf{C}$ are defined by
$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$
here $a, b, c, d \in \mathbf{R}$.

If $a \in \mathbf{R}$, then we identify $a + 0i$ with a. Thus we think of $\mathbf{R}$ as a subset of $\mathbf{C}$. We also usually write $0 + bi$ as bi, and we usually write $0 + 1i$ as i. You should verify that $i^2 = -1$.

The symbol i was first used to denote $\sqrt{-1}$ by Leonhard Euler (1707–1783) in 1777.

With the definitions as above, $\mathbf{C}$ satisfies the usual rules of arithmetic. Specifically, with addition and multiplication defined as above, $\mathbf{C}$ is a field, as you should verify. Thus subtraction and division of complex numbers are defined as in any field.

The field $\mathbf{C}$ cannot be made into an ordered field. However, the useful concept of an absolute value can still be defined on $\mathbf{C}$.

Much of this section may be review for many readers.

6.18 Definition $\operatorname{Re} z$***;*** $\operatorname{Im} z$***; absolute value; limits***

Suppose $z = a + bi$, where a and b are real numbers.

- The *real part* of z, denoted $\operatorname{Re} z$, is defined by $\operatorname{Re} z = a$.
- The *imaginary part* of z, denoted $\operatorname{Im} z$, is defined by $\operatorname{Im} z = b$.
- The *absolute value* of z, denoted $|z|$, is defined by $|z| = \sqrt{a^2 + b^2}$.
- If $z_1, z_2, \ldots \in \mathbf{C}$ and $L \in \mathbf{C}$, then $\lim\limits_{k\to\infty} z_k = L$ means $\lim\limits_{k\to\infty} |z_k - L| = 0$.

For b a real number, the usual definition of $|b|$ as a real number is consistent with the new definition just given of $|b|$ with b thought of as a complex number. Note that if $z_1, z_2, \ldots$ is a sequence of complex numbers and $L \in \mathbf{C}$, then

$$\lim_{k\to\infty} z_k = L \iff \lim_{k\to\infty} \operatorname{Re} z_k = \operatorname{Re} L \text{ and } \lim_{k\to\infty} \operatorname{Im} z_k = \operatorname{Im} L.$$

We will reduce questions concerning measurability and integration of a complex-valued function to the corresponding questions about the real and imaginary parts of the function. We begin this process with the following definition.

6.19 Definition ***measurable complex-valued function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f\colon X \to \mathbf{C}$ is called *$\mathcal{S}$-measurable* if $\operatorname{Re} f$ and $\operatorname{Im} f$ are both $\mathcal{S}$-measurable functions.

See Exercise 5 in this section for two natural conditions that are equivalent to measurability for complex-valued functions.

We will make frequent use of the following result. See Exercise 6 in this section for algebraic combinations of complex-valued measurable functions.

6.20 ***$|f|^p$ is measurable if f is measurable***

Suppose $(X, \mathcal{S})$ is a measurable space, $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function, and $0 < p < \infty$. Then $|f|^p$ is an $\mathcal{S}$-measurable function.

Proof The functions $(\operatorname{Re} f)^2$ and $(\operatorname{Im} f)^2$ are $\mathcal{S}$-measurable because the square of an $\mathcal{S}$-measurable function is measurable (by Example 2.45). Thus the function $(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2$ is $\mathcal{S}$-measurable (because the sum of two $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable by 2.46). Now $\big((\operatorname{Re} f)^2 + (\operatorname{Im} f)^2\big)^{p/2}$ is $\mathcal{S}$-measurable because it is the composition of a continuous function on $[0, \infty)$ and an $\mathcal{S}$-measurable function (see 2.44 and 2.41). In other words, $|f|^p$ is an $\mathcal{S}$-measurable function. ∎

Now we define integration of a complex-valued function by separating the function into its real and imaginary parts.

6.21 Definition ***integral of a complex-valued function***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function with $\int |f| \, d\mu < \infty$ [the collection of such functions is denoted $\mathcal{L}^1(\mu)$]. Then $\int f \, d\mu$ is defined by

$$\int f \, d\mu = \int (\operatorname{Re} f) \, d\mu + i \int (\operatorname{Im} f) \, d\mu.$$

The integral of a complex-valued measurable function is defined above only when the absolute value of the function has a finite integral. In contrast, the integral of every nonnegative measurable function is defined (although the value may be ∞), and if f is real valued then $\int f \, d\mu$ is defined to be $\int f^+ \, d\mu - \int f^- \, d\mu$ if at least one of $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ is finite.

You can easily show that if $f, g\colon X \to \mathbf{C}$ are $\mathcal{S}$-measurable functions such that $\int |f|\, d\mu < \infty$ and $\int |g|\, d\mu < \infty$, then

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu.$$

Similarly, the definition of complex multiplication leads to the conclusion that

$$\int \alpha f\, d\mu = \alpha \int f\, d\mu$$

for all $\alpha \in \mathbf{C}$ (see Exercise 8).

The inequality in the result below concerning integration of complex-valued functions does not follow immediately from the corresponding result for real-valued functions. However, the small trick used in the proof below does give a reasonably simple proof.

6.22 ***bound on the absolute value of an integral***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f|\, d\mu < \infty$. Then

$$\left|\int f\, d\mu\right| \leq \int |f|\, d\mu.$$

Proof The result clearly holds if $\int f\, d\mu = 0$. Thus assume that $\int f\, d\mu \neq 0$.

Let

$$\alpha = \frac{\left|\int f\, d\mu\right|}{\int f\, d\mu}.$$

Then

$$\begin{aligned}
\left|\int f\, d\mu\right| = \alpha \int f\, d\mu &= \int \alpha f\, d\mu \\
&= \int \operatorname{Re}(\alpha f)\, d\mu + i \int \operatorname{Im}(\alpha f)\, d\mu \\
&= \int \operatorname{Re}(\alpha f)\, d\mu \\
&\leq \int |\alpha f|\, d\mu \\
&= \int |f|\, d\mu,
\end{aligned}$$

where the second equality holds by Exercise 8, the fourth equality holds because $\left|\int f\, d\mu\right| \in \mathbf{R}$, the inequality on the fourth line holds because $\operatorname{Re} z \leq |z|$ for every complex number z, and the equality in the last line holds because $|\alpha| = 1$. ■

Because of the result above, the Bounded Convergence Theorem (3.26) and the Dominated Convergence Theorem (3.31) hold if the functions $f_1, f_2, \ldots$ and f in the statements of those theorems are allowed to be complex valued.

We now define the complex conjugate of a complex number.

6.23 Definition ***complex conjugate;*** $\bar{z}$

Suppose $z \in \mathbf{C}$. The *complex conjugate* of $z \in \mathbf{C}$, denoted $\bar{z}$ (pronounced z-bar), is defined by

$$\bar{z} = \operatorname{Re} z - (\operatorname{Im} z)i.$$

For example, if $z = 5 + 7i$ then $\bar{z} = 5 - 7i$. Note that a complex number z is a real number if and only if $z = \bar{z}$.

The next result gives basic properties of the complex conjugate.

6.24 ***properties of complex conjugates***

Suppose $w, z \in \mathbf{C}$. Then

- **product of z and $\bar{z}$**
  $z\,\bar{z} = |z|^2$;
- **sum and difference of z and $\bar{z}$**
  $z + \bar{z} = 2\operatorname{Re} z$ and $z - \bar{z} = 2(\operatorname{Im} z)i$;
- **additivity and multiplicativity of complex conjugate**
  $\overline{w + z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\,\bar{z}$;
- **complex conjugate of complex conjugate**
  $\bar{\bar{z}} = z$;
- **absolute value of complex conjugate**
  $|\bar{z}| = |z|$;
- **integral of complex conjugate of a function**
  $\int \bar{f}\,d\mu = \overline{\int f\,d\mu}$ for every measure μ and every $f \in \mathcal{L}^1(\mu)$.

Proof The first item holds because

$$z\bar{z} = (\operatorname{Re} z + i\operatorname{Im} z)(\operatorname{Re} z - i\operatorname{Im} z) = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = |z|^2.$$

To prove the last item, suppose μ is a measure and $f \in \mathcal{L}^1(\mu)$. Then

$$\begin{aligned}\int \bar{f}\,d\mu = \int (\operatorname{Re} f - i\operatorname{Im} f)\,d\mu &= \int \operatorname{Re} f\,d\mu - i\int \operatorname{Im} f\,d\mu\\ &= \overline{\int \operatorname{Re} f\,d\mu + i\int \operatorname{Im} f\,d\mu}\\ &= \overline{\int f\,d\mu},\end{aligned}$$

as desired.

The straightforward proofs of the remaining items are left to the reader. ■

Vector Spaces and Subspaces

The structure and language of vector spaces will help us focus on certain features of collections of measurable functions. So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation.

6.25 Definition F

From now on, **F** stands for either **R** or **C**.

In the definitions that follow, we use f and g to denote elements of V because in the crucial examples the elements of V are functions from a set X to **F**.

6.26 Definition ***addition; scalar multiplication***

- An *addition* on a set V is a function that assigns an element $f + g \in V$ to each pair of elements $f, g \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\alpha f \in V$ to each $\alpha \in \mathbf{F}$ and each $f \in V$.

Now we are ready to give the formal definition of a vector space.

6.27 Definition ***vector space***

A *vector space* (over **F**) is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity
$f + g = g + f$ for all $f, g \in V$;

associativity
$(f + g) + h = f + (g + h)$ and $(\alpha\beta)f = \alpha(\beta f)$ for all $f, g, h \in V$ and $\alpha, \beta \in \mathbf{F}$;

additive identity
there exists an element $0 \in V$ such that $f + 0 = f$ for all $f \in V$;

additive inverse
for every $f \in V$, there exists $g \in V$ such that $f + g = 0$;

multiplicative identity
$1f = f$ for all $f \in V$;

distributive properties
$\alpha(f + g) = \alpha f + \alpha g$ and $(\alpha + \beta)f = \alpha f + \beta f$ for all $\alpha, \beta \in \mathbf{F}$ and $f, g \in V$.

Most vector spaces that you will encounter are subsets of the vector space $\mathbf{F}^X$ presented in the next example.

6.28 Example *the vector space* $\mathbf{F}^X$

Suppose X is a nonempty set. Let $\mathbf{F}^X$ denote the set of functions from X to $\mathbf{F}$. Addition and scalar multiplication on $\mathbf{F}^X$ are defined as expected: for $f, g \in \mathbf{F}^X$ and $\alpha \in \mathbf{F}$, define

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha\bigl(f(x)\bigr)$$

for $x \in X$. Then, as you should verify, $\mathbf{F}^X$ is a vector space; the additive identity in this vector space is the function $0 \in \mathbf{F}^X$ defined by $0(x) = 0$ for all $x \in X$.

6.29 Example $\mathbf{F}^n$; $\mathbf{F}^{\mathbf{Z}^+}$

Special case of the previous example: if $n \in \mathbf{Z}^+$ and $X = \{1, \dots, n\}$, then $\mathbf{F}^X$ is the familiar space $\mathbf{R}^n$ or $\mathbf{C}^n$, depending upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

Another special case: $\mathbf{F}^{\mathbf{Z}^+}$ is the vector space of all sequences of real numbers or complex numbers, again depending upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

By considering subspaces, we can greatly expand our examples of vector spaces.

6.30 Definition ***subspace***

A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V).

The next result gives the easiest way to check whether a subset of a vector space is a subspace.

6.31 ***conditions for a subspace***

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- **additive identity**
  $0 \in U$;
- **closed under addition**
  $f, g \in U$ implies $f + g \in U$;
- **closed under scalar multiplication**
  $\alpha \in \mathbf{F}$ and $f \in U$ implies $\alpha f \in U$.

Proof If U is a subspace of V, then U satisfies the three conditions above by the definition of vector space.

Conversely, suppose U satisfies the three conditions above. The first condition above ensures that the additive identity of V is in U.

The second condition above ensures that addition makes sense on U. The third condition ensures that scalar multiplication makes sense on U.

If $f \in V$, then $0f = (0+0)f = 0f + 0f$. Adding the additive inverse of $0f$ to both sides of this equation shows that $0f = 0$. Now if $f \in U$, then $(-1)f$ is also in U by the third condition above. Because $f + (-1)f = \bigl(1 + (-1)\bigr)f = 0f = 0$, we see that $(-1)f$ is an additive inverse of f. Hence every element of U has an additive inverse in U.

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V. Thus U is a vector space and hence is a subspace of V. ■

The three conditions in 6.31 usually enable us to determine quickly whether a given subset of V is a subspace of V, as illustrated below. All the examples below except for the first bullet point involve concepts from measure theory.

6.32 Example *subspaces of* $\mathbf{F}^X$

- The set $C([0,1])$ of continuous real-valued functions on $[0,1]$ is a vector space over $\mathbf{R}$ because the sum of two continuous functions is continuous and a constant multiple of a continuous functions is continuous. In other words, $C([0,1])$ is a subspace of $\mathbf{R}^{[0,1]}$.

- Suppose $(X, \mathcal{S})$ is a measurable space. Then the set of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ is a subspace of $\mathbf{F}^X$ because the sum of two $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable and a constant multiple of an $\mathcal{S}$-measurable function is $\mathcal{S}$-measurable.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set $\mathcal{Z}(\mu)$ of $\mathcal{S}$-measurable functions f from X to $\mathbf{F}$ such that $f = 0$ almost everywhere [meaning that $\mu\bigl(\{x \in X : f(x) \neq 0\}\bigr) = 0$] is a vector space over $\mathbf{F}$ because the union of two sets with μ-measure 0 is a set with μ-measure 0 [which implies that $\mathcal{Z}(\mu)$ is closed under addition]. Note that $\mathcal{Z}(\mu)$ is a subspace of $\mathbf{F}^X$.

- Suppose $(X, \mathcal{S})$ is a measurable space. Then the set of bounded measurable functions from X to $\mathbf{F}$ is a subspace of $\mathbf{F}^X$ because the sum of two bounded $\mathcal{S}$-measurable functions is a bounded $\mathcal{S}$-measurable function and a constant multiple of a bounded $\mathcal{S}$-measurable function is a bounded $\mathcal{S}$-measurable function.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set of $\mathcal{S}$-measurable functions f from X to $\mathbf{F}$ such that $\int f \, d\mu = 0$ is a subspace of $\mathbf{F}^X$ because of standard properties of integration.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set $\mathcal{L}^1(\mu)$ of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ such that $\int |f| \, d\mu < \infty$ is a subspace of $\mathbf{F}^X$ [we are now redefining $\mathcal{L}^1(\mu)$ to allow for the possibility that $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$]. The set $\mathcal{L}^1(\mu)$ is closed under addition and scalar multiplication because $\int |f+g| \, d\mu \leq \int |f| \, d\mu + \int |g| \, d\mu$ and $\int |\alpha f| \, d\mu = |\alpha| \int |f| \, d\mu$.

- The set ℓ^1 of all sequences $(a_1, a_2, \ldots)$ of elements of $\mathbf{F}$ such that $\sum_{k=1}^{\infty} |a_k| < \infty$ is a subspace of $\mathbf{F}^{\mathbf{Z}^+}$. Note that ℓ^1 is a special case of the example in the previous bullet point (take μ to be counting measure on $\mathbf{Z}^+$).

EXERCISES 6B

1 Show that if $a, b \in \mathbf{R}$ with $a + bi \neq 0$, then

$$\frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

2 Suppose $z \in \mathbf{C}$. Prove that

$$\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} \leq |z| \leq \sqrt{2}\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}.$$

3 Suppose $z \in \mathbf{C}$. Prove that $\dfrac{|\operatorname{Re} z| + |\operatorname{Im} z|}{\sqrt{2}} \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$.

4 Suppose $w, z \in \mathbf{C}$. Prove that $|wz| = |w|\,|z|$ and $|w + z| \leq |w| + |z|$.

5 Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to \mathbf{C}$ is a complex-valued function. For conditions (b) and (c) below, identify $\mathbf{C}$ with $\mathbf{R}^2$. Prove that the following are equivalent:

(a) f is $\mathcal{S}$-measurable.
(b) $f^{-1}(G) \in \mathcal{S}$ for every open set G in $\mathbf{R}^2$.
(c) $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \in \mathcal{B}_2$.

6 Suppose $(X, \mathcal{S})$ is a measurable space and $f, g\colon X \to \mathbf{C}$ are $\mathcal{S}$-measurable. Prove that

(a) $f + g$, $f - g$, and fg are $\mathcal{S}$-measurable functions;
(b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

7 Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{C}$. Suppose $\lim\limits_{k\to\infty} f_k(x)$ exists for each $x \in X$. Define $f\colon X \to \mathbf{C}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Prove that f is an $\mathcal{S}$-measurable function.

8 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f|\, d\mu < \infty$. Prove that if $\alpha \in \mathbf{C}$, then

$$\int \alpha f\, d\mu = \alpha \int f\, d\mu.$$

9 Suppose V is a vector space. Show that the intersection of every collection of subspaces of V is a subspace of V.

10 Suppose V and W are vector spaces. Define $V \times W$ by

$$V \times W = \{(f, g) : f \in V \text{ and } g \in W\}.$$

Define addition and scalar multiplication on $V \times W$ by

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2) \quad \text{and} \quad \alpha(f, g) = (\alpha f, \alpha g).$$

Prove that $V \times W$ is a vector space with these operations.

6C Normed Vector Spaces

Norms and Complete Norms

This section begins with a crucial definition.

6.33 Definition ***norm; normed vector space***

A *norm* on a vector space V (over $\mathbf{F}$) is a function $\|\cdot\|: V \to [0, \infty)$ such that

- $\|f\| = 0$ if and only if $f = 0$ (**positive definite**);
- $\|\alpha f\| = |\alpha| \, \|f\|$ for all $\alpha \in \mathbf{F}$ and $f \in V$ (**homogeneity**);
- $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in V$ (**triangle inequality**).

A *normed vector space* is a pair $(V, \|\cdot\|)$, where V is a vector space and $\|\cdot\|$ is a norm on V.

6.34 Example *norms*

- Suppose $n \in \mathbf{Z}^+$. Define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\|_1 = |a_1| + \cdots + |a_n|$$
and
$$\|(a_1, \dots, a_n)\|_\infty = \max\{|a_1|, \dots, |a_n|\}.$$
Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbf{F}^n$, as you should verify.

- On ℓ^1 (see the last bullet point in Example 6.32 for the definition of ℓ^1), define $\|\cdot\|_1$ by
$$\|(a_1, a_2, \dots)\|_1 = \sum_{k=1}^{\infty} |a_k|.$$
Then $\|\cdot\|_1$ is a norm on ℓ^1, as you should verify.

- Suppose X is a nonempty set and $b(X)$ is the subspace of $\mathbf{F}^X$ consisting of the bounded functions from X to $\mathbf{F}$. For f a bounded function from X to $\mathbf{F}$, define $\|f\|$ by
$$\|f\| = \sup\{|f(x)| : x \in X\}.$$
Then $\|\cdot\|$ is a norm on $b(X)$, as you should verify.

- Let $C([0,1])$ denote the vector space of continuous functions from the interval $[0,1]$ to $\mathbf{F}$. Define $\|\cdot\|$ on $C([0,1])$ by
$$\|f\| = \int_0^1 |f|.$$
Then $\|\cdot\|$ is a norm on $C([0,1])$, as you should verify.

Sometimes examples that do not satisfy a definition help you gain understanding.

6.35 Example *not norms*

- Let $\mathcal{L}^1(\mathbf{R})$ denote the vector space of Borel (or Lebesgue) measurable functions $f \colon \mathbf{R} \to \mathbf{F}$ such that $\int |f|\, d\lambda < \infty$, where λ is Lebesgue measure on $\mathbf{R}$. Define $\|\cdot\|_1$ on $\mathcal{L}^1(\mathbf{R})$ by
$$\|f\|_1 = \int |f|\, d\lambda.$$
Then $\|\cdot\|_1$ satisfies the homogeneity condition and the triangle inequality on $\mathcal{L}^1(\mathbf{R})$, as you should verify. However, $\|\cdot\|_1$ is not a norm on $\mathcal{L}^1(\mathbf{R})$ because the positive definite condition is not satisfied. Specifically, if E is a nonempty Borel subset of $\mathbf{R}$ with Lebesgue measure 0 (for example, E might consist of a single element of $\mathbf{R}$), then $\|\chi_E\|_1 = 0$ but $\chi_E \neq 0$. In the next chapter, we will discuss a modification of $\mathcal{L}^1(\mathbf{R})$ that removes this problem.

- If $n \in \mathbf{Z}^+$ and $\|\cdot\|$ is defined on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\| = |a_1|^{1/2} + \cdots + |a_n|^{1/2},$$
then $\|\cdot\|$ satisfies the positive definite condition and the triangle inequality (as you should verify). However, $\|\cdot\|$ as defined above is not a norm because it does not satisfy the homogeneity condition.

- If $\|\cdot\|_{1/2}$ is defined on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\|_{1/2} = \bigl(|a_1|^{1/2} + \cdots + |a_n|^{1/2}\bigr)^2,$$
then $\|\cdot\|_{1/2}$ satisfies the positive definite condition and the homogeneity condition. However, if $n > 1$ then $\|\cdot\|_{1/2}$ is not a norm on $\mathbf{F}^n$ because the triangle inequality is not satisfied (as you should verify).

The next result shows that every normed vector space is also a metric space in a natural fashion.

6.36 *normed vector spaces are metric spaces*

Suppose $(V, \|\cdot\|)$ is a normed vector space. Define $d \colon V \times V \to [0, \infty)$ by
$$d(f, g) = \|f - g\|.$$
Then d is a metric on V.

Proof Suppose $f, g, h \in V$. Then
$$\begin{aligned} d(f, h) = \|f - h\| &= \|(f - g) + (g - h)\| \\ &\leq \|f - g\| + \|g - h\| \\ &= d(f, g) + d(g, h). \end{aligned}$$
Thus the triangle inequality requirement for a metric is satisfied. The verification of the other required properties for a metric are left to the reader. ∎

From now on, all metric space notions in the context of a normed vector space should be interpreted with respect to the metric introduced in the previous result. However, usually there is no need to introduce the metric d explicitly—just use the norm of the difference of two elements. For example, suppose $(V, \|\cdot\|)$ is a normed vector space, $f_1, f_2, \dots$ is a sequence in V, and $f \in V$. Then in the context of a normed vector space, the definition of limit (6.8) becomes the following statement:

$$\lim_{k\to\infty} f_k = f \text{ means } \lim_{k\to\infty} \|f_k - f\| = 0.$$

As another example, in the context of a normed vector space, the definition of a Cauchy sequence (6.12) becomes the following statement:

> A sequence $f_1, f_2, \dots$ in a normed vector space $(V, \|\cdot\|)$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $\|f_j - f_k\| < \varepsilon$ for all integers $j \geq n$ and $k \geq n$.

Every sequence in a normed vector space that has a limit is a Cauchy sequence (see 6.13). Normed vector spaces that satisfy the converse have a special name.

6.37 Definition ***Banach space***

A complete normed vector space is called a *Banach space*.

In other words, a normed vector space V is a Banach space if every Cauchy sequence in V converges to some element of V.

The verifications of the assertions in Examples 6.38 and 6.39 below are left to the reader as exercises.

In a slight abuse of terminology, we often refer to a normed vector space V without mentioning the norm $\|\cdot\|$. When that happens, you should assume that a norm $\|\cdot\|$ lurks nearby, even if it is not explicitly displayed.

6.38 Example *Banach spaces*

- The vector space $C([0,1])$ with the norm defined by $\|f\| = \sup_{[0,1]} |f|$ is a Banach space.
- The vector space ℓ^1 with the norm defined by $\|(a_1, a_2, \dots)\|_1 = \sum_{k=1}^{\infty} |a_k|$ is a Banach space.

6.39 Example *not a Banach space*

- The vector space $C([0,1])$ with the norm defined by $\|f\| = \int_0^1 |f|$ is not a Banach space.
- The vector space ℓ^1 with the norm defined by $\|(a_1, a_2, \dots)\|_\infty = \sup_{k\in\mathbf{Z}^+} |a_k|$ is not a Banach space.

6.40 Definition ***infinite sum in a normed vector space***

Suppose $g_1, g_2, \dots$ is a sequence in a normed vector space V. Then $\sum_{k=1}^{\infty} g_k$ is defined by

$$\sum_{k=1}^{\infty} g_k = \lim_{n\to\infty} \sum_{k=1}^{n} g_k$$

if this limit exists, in which case the infinite series is said to *converge*.

Recall from your calculus course that if $a_1, a_2, \dots$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges. The next result states that the analogous property for normed vector spaces characterizes Banach spaces.

6.41 $\bigl(\sum_{k=1}^{\infty} \|g_k\| < \infty \implies \sum_{k=1}^{\infty} g_k$ ***converges***$\bigr) \iff$ ***Banach space***

Suppose V is a normed vector space. Then V is a Banach space if and only if $\sum_{k=1}^{\infty} g_k$ converges for every sequence $g_1, g_2, \dots$ in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$.

Proof First suppose V is a Banach space. Suppose $g_1, g_2, \dots$ is a sequence in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$. Suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $\sum_{m=n}^{\infty} \|g_m\| < \varepsilon$. For $j \in \mathbf{Z}^+$, let f_j denote the partial sum defined by

$$f_j = g_1 + \cdots + g_j.$$

If $k > j \geq n$, then

$$\begin{aligned} \|f_k - f_j\| &= \|g_{j+1} + \cdots + g_k\| \\ &\leq \|g_{j+1}\| + \cdots + \|g_k\| \\ &\leq \sum_{m=n}^{\infty} \|g_m\| \\ &< \varepsilon. \end{aligned}$$

Thus $f_1, f_2, \dots$ is a Cauchy sequence in V. Because V is a Banach space, we conclude that $f_1, f_2, \dots$ converges to some element of V, which is precisely what it means for $\sum_{k=1}^{\infty} g_k$ to converge, completing one direction of the proof.

To prove the other direction, suppose $\sum_{k=1}^{\infty} g_k$ converges for every sequence $g_1, g_2, \dots$ in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$. Suppose $f_1, f_2, \dots$ is a Cauchy sequence in V. We want to prove that $f_1, f_2, \dots$ converges to some element of V. It suffices to show that some subsequence of $f_1, f_2, \dots$ converges (by Exercise 14 in Section 6A). Dropping to a subsequence (but not relabeling) and setting $f_0 = 0$, we can assume that

$$\sum_{k=1}^{\infty} \|f_k - f_{k-1}\| < \infty.$$

Hence $\sum_{k=1}^{\infty} (f_k - f_{k-1})$ converges. The partial sum of this series after n terms is f_n. Thus $\lim_{n\to\infty} f_n$ exists, completing the proof. ∎

Bounded Linear Maps

When dealing with two or more vector spaces, as in the definition below, assume that the vector spaces are over the same field (either **R** or **C**, but denoted in this book as **F** to give us the flexibility to consider both cases).

The notation Tf, in addition to the standard functional notation $T(f)$, is often used when considering linear maps, which we now define.

6.42 Definition ***linear map***

Suppose V and W are vector spaces. A function $T\colon V \to W$ is called *linear* if

- $T(f+g) = Tf + Tg$ for all $f, g \in V$;
- $T(\alpha f) = \alpha Tf$ for all $\alpha \in \mathbf{F}$ and $f \in V$.

A linear function is often called a *linear map*.

The set of linear maps from a vector space V to a vector space W is itself a vector space, using the usual operations of addition and scalar multiplication of functions. Most attention in analysis focuses on the subset of bounded linear functions, defined below, which we will see is itself a normed vector space.

In the next definition, we have two normed vector spaces, V and W, which may have different norms. However, we use the same notation $\|\cdot\|$ for both norms (and for the norm of a linear map from V to W) because the context makes the meaning clear. For example, in the definition below, f is in V and thus $\|f\|$ refers to the norm in V. Similarly, $Tf \in W$ and thus $\|Tf\|$ refers to the norm in W.

6.43 Definition ***bounded linear map; $\|T\|$; $\mathcal{B}(V, W)$***

Suppose V and W are normed vector spaces and $T\colon V \to W$ is a linear map.

- The norm of T, denoted $\|T\|$, is defined by
$$\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| \leq 1\}.$$
- T is called *bounded* if $\|T\| < \infty$.
- The set of bounded linear maps from V to W is denoted $\mathcal{B}(V, W)$.

6.44 Example *bounded linear map*

Let $C([0,3])$ be the normed vector space of continuous functions from $[0,3]$ to $\mathbf{F}$, with $\|f\| = \sup\limits_{[0,3]}|f|$. Define $T\colon C([0,3]) \to C([0,3])$ by

$$(Tf)(x) = x^2 f(x).$$

Then T is a bounded linear map and $\|T\| = 9$, as you should verify.

6.45 Example *linear map that is not bounded*

Let V be the normed vector space of sequences $(a_1, a_2, \dots)$ of elements of $\mathbf{F}$ such that $a_k = 0$ for all but finitely many $k \in \mathbf{Z}^+$, with $\|(a_1, a_2, \dots)\|_\infty = \max_{k \in \mathbf{Z}^+} |a_k|$. Define $T \colon V \to V$ by

$$T(a_1, a_2, a_3, \dots) = (a_1, 2a_2, 3a_3, \dots).$$

Then T is a linear map that is not bounded, as you should verify.

The next result shows that if V and W are normed vector spaces, then $\mathcal{B}(V, W)$ is a normed vector space with the norm defined above.

6.46 $\|\cdot\|$ ***is a norm on*** $\mathcal{B}(V, W)$

Suppose V and W are normed vector spaces. Then $\|S + T\| \leq \|S\| + \|T\|$ and $\|\alpha T\| = |\alpha| \, \|T\|$ for all $S, T \in \mathcal{B}(V, W)$ and all $\alpha \in \mathbf{F}$. Furthermore, the function $\|\cdot\|$ is a norm on $\mathcal{B}(V, W)$.

Proof Suppose $S, T \in \mathcal{B}(V, W)$. then

$$\begin{aligned} \|S + T\| &= \sup\{\|(S + T)f\| : f \in V \text{ and } \|f\| \leq 1\} \\ &\leq \sup\{\|Sf\| + \|Tf\| : f \in V \text{ and } \|f\| \leq 1\} \\ &\leq \sup\{\|Sf\| : f \in V \text{ and } \|f\| \leq 1\} \\ &\qquad\qquad + \sup\{\|Tf\| : f \in V \text{ and } \|f\| \leq 1\} \\ &= \|S\| + \|T\|. \end{aligned}$$

The inequality above shows that $\|\cdot\|$ satisfies the triangle inequality on $\mathcal{B}(V, W)$. The verification of the other properties required for a normed vector space is left to the reader. ■

Be sure that you are comfortable using all four equivalent formulas for $\|T\|$ shown in Exercise 16. For example, you should often think of $\|T\|$ as the smallest number such that $\|Tf\| \leq \|T\| \, \|f\|$ for all f in the domain of T.

Note that in the next result, the hypothesis requires W to be a Banach space but there is no requirement for V to be a Banach space.

6.47 $\mathcal{B}(V, W)$ ***is a Banach space if*** W ***is a Banach space***

Suppose V is a normed vector space and W is a Banach space. Then $\mathcal{B}(V, W)$ is a Banach space.

Proof Suppose $T_1, T_2, \dots$ is a Cauchy sequence in $\mathcal{B}(V, W)$. If $f \in V$, then

$$\|T_j f - T_k f\| \leq \|T_j - T_k\| \, \|f\|,$$

which implies that $T_1 f, T_2 f, \dots$ is a Cauchy sequence in W. Because W is a Banach space, this implies that $T_1 f, T_2 f, \dots$ has a limit in W, which we call Tf.

We have now defined a function $T\colon V \to W$. The reader should verify that T is a linear map. Clearly

$$\begin{aligned}\|Tf\| &\le \sup\{\|T_k f\| : k \in \mathbf{Z}^+\}\\ &\le \bigl(\sup\{\|T_k\| : k \in \mathbf{Z}^+\}\bigr)\|f\|\end{aligned}$$

for each $f \in V$. The last supremum above is finite because every Cauchy sequence is bounded (see Exercise 4). Thus $T \in \mathcal{B}(V, W)$.

We still need to show that $\lim_{k\to\infty}\|T_k - T\| = 0$. To do this, suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $\|T_j - T_k\| < \varepsilon$ for all $j \ge n$ and $k \ge n$. Suppose $j \ge n$ and suppose $f \in V$. Then

$$\begin{aligned}\|(T_j - T)f\| &= \lim_{k\to\infty}\|T_j f - T_k f\|\\ &\le \varepsilon\|f\|.\end{aligned}$$

Thus $\|T_j - T\| \le \varepsilon$, completing the proof. ■

The next result shows that the phrase *bounded linear map* means the same as the phrase *continuous linear map*.

6.48 ***continuity is equivalent to boundedness for linear maps***

A linear map from one normed vector space to another normed vector space is continuous if and only if it is bounded.

Proof Suppose V and W are normed vector spaces and $T\colon V \to W$ is linear.

First suppose T is not bounded. Thus there exists a sequence $f_1, f_2, \ldots$ in V such that $\|f_k\| \le 1$ for each $k \in \mathbf{Z}^+$ and $\|Tf_k\| \to \infty$ as $k \to \infty$. Hence

$$\lim_{k\to\infty}\frac{f_k}{\|Tf_k\|} = 0 \quad\text{and}\quad T\Bigl(\frac{f_k}{\|Tf_k\|}\Bigr) = \frac{Tf_k}{\|Tf_k\|} \not\to 0,$$

where the nonconvergence to 0 holds because $Tf_k/\|Tf_k\|$ has norm 1 for every $k \in \mathbf{Z}^+$. The displayed line above implies that T is not continuous, completing the proof in one direction.

To prove the other direction, now suppose T is bounded. Suppose $f \in V$ and $f_1, f_2, \ldots$ is a sequence in V such that $\lim_{k\to\infty} f_k = f$. Then

$$\begin{aligned}\|Tf_k - Tf\| &= \|T(f_k - f)\|\\ &\le \|T\|\,\|f_k - f\|.\end{aligned}$$

Thus $\lim_{k\to\infty} Tf_k = Tf$. Hence T is continuous, completing the proof in the other direction. ■

Exercise 18 gives several additional equivalent conditions for a linear map to be continuous.

EXERCISES 6C

1 Show that the map $f \mapsto \|f\|$ from a normed vector space V to $\mathbf{F}$ is continuous (where the norm on $\mathbf{F}$ is the usual absolute value).

2 Prove that if V is a normed vector space, $f \in V$, and $r > 0$, then

$$\overline{B(f,r)} = \overline{B}(f,r).$$

3 Show that the functions defined in the last two bullet points of Example 6.35 are not norms.

4 Prove that each Cauchy sequence in a normed vector space is bounded (meaning that there is a real number that is greater than the norm of every element in the Cauchy sequence).

5 Show that if $n \in \mathbf{Z}^+$, then $\mathbf{F}^n$ is a Banach space with both the norms used in the first bullet point of Example 6.34.

6 Suppose X is a nonempty set and $b(X)$ is the vector space of bounded functions from X to $\mathbf{F}$. Prove that if $\|\cdot\|$ is defined on $b(X)$ by $\|f\| = \sup_X |f|$, then $b(X)$ is a Banach space.

7 Show that ℓ^1 with the norm defined by $\|(a_1, a_2, \ldots)\|_\infty = \sup_{k \in \mathbf{Z}^+} |a_k|$ is not a Banach space.

8 Show that ℓ^1 with the norm defined by $\|(a_1, a_2, \ldots)\|_1 = \sum_{k=1}^{\infty} |a_k|$ is a Banach space.

9 Show that the vector space $C([0,1])$ of continuous functions from $[0,1]$ to $\mathbf{F}$ with the norm defined by $\|f\| = \int_0^1 |f|$ is not a Banach space.

10 Suppose U is a subspace of a normed vector space V such that some open ball of V is contained in U. Prove that $U = V$.

11 Prove that the only subsets of a normed vector space V that are both open and closed are $\varnothing$ and V.

12 Suppose V is a normed vector space. Prove that the closure of each subspace of V is a subspace of V.

13 Suppose U is a normed vector space. Let d be the metric on U defined by $d(f,g) = \|f - g\|$ for $f, g \in U$. Let V be the complete metric space constructed in Exercise 16 in Section 6A.

(a) Show that the set V is a vector space under natural operations of addition and scalar multiplication.

(b) Show that there is a natural way to make V into a normed vector space and that with this norm, V is a Banach space.

(c) Explain why (b) shows that every normed vector space is a subspace of some Banach space.

14 Suppose U is a subspace of a normed vector space V. Suppose also that W is a Banach space and $S\colon U \to W$ is a bounded linear map.

(a) Prove that there exists a unique continuous function $T\colon \overline{U} \to W$ such that $T|_U = S$.

(b) Prove that the function T in part (a) is a bounded linear map from $\overline{U}$ to W and $\|T\| = \|S\|$.

(c) Give an example to show that part (a) can fail if the assumption that W is a Banach space is replaced by the assumption that W is a normed vector space.

15 For readers familiar with the quotient of a vector space and a subspace: Suppose V is a normed vector space and U is a subspace of V. Define $\|\cdot\|$ on V/U by

$$\|f + U\| = \inf\{\|f + g\| : g \in U\}.$$

(a) Prove that $\|\cdot\|$ is a norm on V/U if and only if U is a closed subspace of V.

(b) Prove that if V is a Banach space and U is a closed subspace of V, then V/U (with the norm defined above) is a Banach space.

(c) Prove that if U is a Banach space (with the norm it inherits from V) and V/U is a Banach space (with the norm defined above), then V is a Banach space.

16 Suppose V and W are normed vector spaces with $V \neq \{0\}$ and $T\colon V \to W$ is a linear map.

(a) Show that $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| < 1\}$.

(b) Show that $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| = 1\}$.

(c) Show that $\|T\| = \inf\{c \in [0, \infty) : \|Tf\| \leq c\|f\| \text{ for all } f \in V\}$.

(d) Show that $\|T\| = \sup\Bigl\{\dfrac{\|Tf\|}{\|f\|} : f \in V \text{ and } f \neq 0\Bigr\}$.

17 Suppose U, V, and W are normed vector spaces and $T\colon U \to V$ and $S\colon V \to W$ are linear. Prove that $\|S \circ T\| \leq \|S\|\,\|T\|$.

18 Suppose V and W are normed vector spaces and $T\colon V \to W$ is a linear map. Prove that the following are equivalent:

(a) T is bounded.

(b) There exists $f \in V$ such that T is continuous at f.

(c) T is uniformly continuous (which means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Tf - Tg\| < \varepsilon$ for all $f, g \in V$ with $\|f - g\| < \delta$).

(d) $T^{-1}\bigl(B(0, r)\bigr)$ is an open subset of V for some $r > 0$.

6D Linear Functionals

Bounded Linear Functionals

Linear maps into the scalar field $\mathbf{F}$ are so important that they get a special name.

6.49 Definition *linear functional*

A *linear functional* on a vector space V is a linear map from V to $\mathbf{F}$.

When we think of the scalar field $\mathbf{F}$ as a normed vector space, as in the next example, the norm $\|z\|$ of a number $z \in \mathbf{F}$ is always intended to be just the usual absolute value $|z|$. This norm makes $\mathbf{F}$ a Banach space.

6.50 Example *linear functional*

Let V be the vector space of sequences $(a_1, a_2, \ldots)$ of elements of $\mathbf{F}$ such that $a_k = 0$ for all but finitely many $k \in \mathbf{Z}^+$. Define $\varphi\colon V \to \mathbf{F}$ by

$$\varphi(a_1, a_2, \ldots) = \sum_{k=1}^{\infty} a_k.$$

Then φ is a linear functional on V.

- If we make V a normed vector space with the norm $\|(a_1, a_2, \ldots)\|_1 = \sum_{k=1}^{\infty} |a_k|$, then φ is a bounded linear functional on V, as you should verify.
- If we make V a normed vector space with the norm $\|(a_1, a_2, \ldots)\|_\infty = \max_{k \in \mathbf{Z}^+} |a_k|$, then φ is not a bounded linear functional on V, as you should verify.

6.51 Definition ***null space;*** null T

Suppose V and W are vector spaces and $T\colon V \to W$ is a linear map. Then the *null space* of T is denoted by null T and is defined by

$$\operatorname{null} T = \{f \in V : Tf = 0\}.$$

If T is a linear map on a vector space V, then null T is a subspace of V, as you should verify. If T is a continuous linear map from a normed vector space V to a normed vector space W, then null T is a closed subspace of V because $\operatorname{null} T = T^{-1}(\{0\})$ and the inverse image of the closed set $\{0\}$ is closed [by 6.11(d)].

*The term **kernel** is also used in the mathematics literature with the same meaning as **null space**. This book uses **null space** instead of **kernel** because **null space** better captures the connection with* 0.

The converse of the last sentence fails, because a linear map between normed vector spaces can have a closed null space but not be continuous. For example, the linear map in 6.45 has a closed null space (equal to $\{0\}$) but it is not continuous.

However, the next result states that for linear functionals, as opposed to more general linear maps, having a closed null space is equivalent to continuity.

6.52 *bounded linear functionals*

Suppose V is a normed vector space and $\varphi\colon V \to \mathbf{F}$ is a linear functional that is not identically 0. Then the following are equivalent:

(a) φ is a bounded linear functional.

(b) φ is a continuous linear functional.

(c) $\operatorname{null}\varphi$ is a closed subspace of V.

(d) $\overline{\operatorname{null}\varphi} \neq V$.

Proof The equivalence of (a) and (b) is just a special case of 6.48.

To prove that (b) implies (c), suppose φ is a continuous linear functional. Then $\operatorname{null}\varphi$, which is the inverse image of the closed set $\{0\}$, is a closed subset of V by 6.11(d). Thus (b) implies (c).

To prove that (c) implies (a), we will show that the negation of (a) implies the negation of (c). Thus suppose φ is not bounded. Thus there is a sequence $f_1, f_2, \ldots$ in V such that $\|f_k\| \leq 1$ and $|\varphi(f_k)| \geq k$ for each $k \in \mathbf{Z}^+$. Now

$$\frac{f_1}{\varphi(f_1)} - \frac{f_k}{\varphi(f_k)} \in \operatorname{null}\varphi$$

for each $k \in \mathbf{Z}^+$ and

$$\lim_{k\to\infty}\Bigl(\frac{f_1}{\varphi(f_1)} - \frac{f_k}{\varphi(f_k)}\Bigr) = \frac{f_1}{\varphi(f_1)}.$$

This proof makes major use of dividing by expressions of the form $\varphi(f)$, which would not make sense for a linear mapping into a vector space other than $\mathbf{F}$.

Clearly

$$\varphi\Bigl(\frac{f_1}{\varphi(f_1)}\Bigr) = 1 \text{ and thus } \frac{f_1}{\varphi(f_1)} \notin \operatorname{null}\varphi.$$

The last three displayed items imply that $\operatorname{null}\varphi$ is not closed, completing the proof that the negation of (a) implies the negation of (c). Thus (c) implies (a).

We now know that (a), (b), and (c) are equivalent to each other.

Using the hypothesis that φ is not identically 0, we see that (c) implies (d). To complete the proof, we need only show that (d) implies (c), which we will do by showing that the negation of (c) implies the negation of (d). Thus suppose $\operatorname{null}\varphi$ is not a closed subspace of V. Because $\operatorname{null}\varphi$ is a subspace of V, we know that $\overline{\operatorname{null}\varphi}$ is also a subspace of V (see Exercise 12 in Section 6C). Let $f \in \overline{\operatorname{null}\varphi} \setminus \operatorname{null}\varphi$. Suppose $g \in V$. Then

$$g = \Bigl(g - \frac{\varphi(g)}{\varphi(f)}f\Bigr) + \frac{\varphi(g)}{\varphi(f)}f.$$

The term in large parentheses above is in $\operatorname{null}\varphi$ and hence is in $\overline{\operatorname{null}\varphi}$. The term above following the plus sign is a scalar multiple of f and thus is in $\overline{\operatorname{null}\varphi}$. Because the equation above writes g as the sum of two elements of $\overline{\operatorname{null}\varphi}$, we conclude that $g \in \overline{\operatorname{null}\varphi}$. Hence we have shown that $V = \overline{\operatorname{null}\varphi}$, completing the proof that the negation of (c) implies the negation of (d). ■

Discontinuous Linear Functionals

The second bullet point in Example 6.50 shows that there exists a discontinuous linear functional on a certain normed vector space. Our next major goal is to show that every infinite-dimensional normed vector space has a discontinuous linear functional (see 6.62). Thus infinite-dimensional normed vector spaces behave in this respect much differently from $\mathbf{F}^n$, where all linear functionals are continuous (see Exercise 4).

We need to extend the notion of a basis of a finite-dimensional vector space to an infinite-dimensional context. In a finite-dimensional vector space, we might consider a basis of the form $e_1, \dots, e_n$, where $n \in \mathbf{Z}^+$ and each e_k is an element of our vector space. We can think of the list $e_1, \dots, e_n$ as a function from $\{1, \dots, n\}$ to our vector space, with the value of this function at $k \in \{1, \dots, n\}$ denoted by e_k with a subscript k instead of by the usual functional notation $e(k)$. To generalize, in the next definition we allow $\{1, \dots, n\}$ to be replaced by an arbitrary set that might not be a finite set.

6.53 Definition ***family***

A *family* $\{e_k\}_{k \in \Gamma}$ in a set V is a function e from a set Γ to V, with the value of the function e at $k \in \Gamma$ denoted by e_k.

Even though a family in V is a function mapping into V and thus is not a subset of V, the set terminology and the bracket notation $\{e_k\}_{k \in \Gamma}$ are useful, and the range of a family in V really is a subset of V.

We now restate some basic linear algebra concepts, but in the context of vector spaces that might be infinite-dimensional. Note that only finite sums appear in the definition below, even though we might be working with an infinite family.

6.54 Definition ***linearly independent; span; basis***

Suppose $\{e_k\}_{k \in \Gamma}$ is a family in a vector space V.

- $\{e_k\}_{k \in \Gamma}$ is called *linearly independent* if there does not exist a finite nonempty subset Ω of Γ and a family $\{\alpha_j\}_{j \in \Omega}$ in $\mathbf{F} \setminus \{0\}$ such that $\sum_{j \in \Omega} \alpha_j e_j = 0$.

- The span of $\{e_k\}_{k \in \Gamma}$ is denoted by $\operatorname{span}\{e_k\}_{k \in \Gamma}$ and is defined to be the set of all sums of the form
$$\sum_{j \in \Omega} \alpha_j e_j,$$
where Ω is a finite subset of Γ and $\{\alpha_j\}_{j \in \Omega}$ is a family in $\mathbf{F}$.

- A vector space V is called *finite-dimensional* if there exists a finite set Γ and a family $\{e_k\}_{k \in \Gamma}$ in V such that $\operatorname{span}\{e_k\}_{k \in \Gamma} = V$.

- A vector space is called *infinite-dimensional* if it is not finite-dimensional.

- A family in V is called a *basis* of V if it is linearly independent and its span equals V.

For example, $\{x^n\}_{n\in\{0,1,2,\ldots\}}$ is a basis of the vector space of polynomials.

The term Hamel basis *is sometimes used to denote what has been called a* basis *here. The use of the term* Hamel basis *emphasizes that only finite sums are under consideration.*

Our definition of span does not take advantage of the possibility of summing an infinite number of elements in contexts where a notion of limit exists (as is the case in normed vector spaces). When we get to Hilbert spaces in Chapter 8, we consider another kind of basis that does involve infinite sums. As we will soon see, the kind of basis as defined here is just what we need to produce discontinuous linear functionals.

Now we introduce terminology that will be needed in our proof that every vector space has a basis.

No one has ever produced a concrete example of a basis of an infinite-dimensional Banach space.

6.55 Definition ***maximal element***

Suppose $\mathcal{A}$ is a collection of subsets of a set V. A set $\Gamma \in \mathcal{A}$ is called a *maximal element* of $\mathcal{A}$ if there does not exist $\Gamma' \in \mathcal{A}$ such that $\Gamma \subsetneq \Gamma'$.

6.56 Example *maximal elements*

For $k \in \mathbf{Z}$, let $k\mathbf{Z}$ denote the set of integer multiples of k; thus $k\mathbf{Z} = \{km : m \in \mathbf{Z}\}$. Let $\mathcal{A}$ be the collection of subsets of $\mathbf{Z}$ defined by $\mathcal{A} = \{k\mathbf{Z} : k = 2, 3, 4, \ldots\}$. Suppose $k \in \mathbf{Z}^+$. Then $k\mathbf{Z}$ is a maximal element of $\mathcal{A}$ if and only if k is a prime number, as you should verify.

A subset Γ of a vector space V can be thought of as a family in V by considering $\{e_f\}_{f\in\Gamma}$, where $e_f = f$. With this convention, the next result shows that the bases of V are exactly the maximal elements among the collection of linearly independent subsets of V.

6.57 ***bases as maximal elements***

Suppose V is a vector space. Then a subset of V is a basis of V if and only if it is a maximal element of the collection of linearly independent subsets of V.

Proof Suppose Γ is a linearly independent subset of V.

First suppose also that Γ is a basis of V. If $f \in V$ but $f \notin \Gamma$, then $f \in \operatorname{span}\Gamma$, which implies that $\Gamma \cup \{f\}$ is not linearly independent. Thus Γ is a maximal element among the collection of linearly independent subsets of V, completing one direction of the proof.

To prove the other direction, suppose now that Γ is a maximal element of the collection of linearly independent subsets of V. If $f \in V$ but $f \notin \operatorname{span}\Gamma$, then $\Gamma \cup \{f\}$ is linearly independent, which would contradict the maximality of Γ among the collection of linearly independent subsets of V. Thus $\operatorname{span}\Gamma = V$, which means that Γ is a basis of V, completing the proof in the other direction. ∎

The notion of a chain plays a key role in our next result.

6.58 Definition ***chain***

A collection $\mathcal{C}$ of subsets of a set V is called a *chain* if $\Omega, \Gamma \in \mathcal{C}$ implies $\Omega \subset \Gamma$ or $\Gamma \subset \Omega$.

6.59 Example *chains*

- The collection $\mathcal{C} = \{4\mathbf{Z}, 6\mathbf{Z}\}$ of subsets of $\mathbf{Z}$ is not a chain because neither of the sets $4\mathbf{Z}$ or $6\mathbf{Z}$ is a subset of the other.
- The collection $\mathcal{C} = \{2^n\mathbf{Z} : n \in \mathbf{Z}^+\}$ of subsets of $\mathbf{Z}$ is a chain because if $m, n \in \mathbf{Z}^+$, then $2^m\mathbf{Z} \subset 2^n\mathbf{Z}$ or $2^n\mathbf{Z} \subset 2^m\mathbf{Z}$.

Zorn's Lemma is named in honor of Max Zorn (1906–1993), who published a paper containing the result in 1935, when he had a postdoctoral position at Yale.

The next result follows from the Axiom of Choice, although it is not as intuitively believable as the Axiom of Choice. Because the techniques used to prove the next result are so different from techniques used elsewhere in this book, the reader is asked either to accept this result without proof or find one of the good proofs available via the internet or in other books. The version of Zorn's Lemma stated here is simpler than the standard more general version, but this version is all that we need.

6.60 ***Zorn's Lemma***

Suppose V is a set and $\mathcal{A}$ is a collection of subsets of V with the property that the union of all the sets in $\mathcal{C}$ is in $\mathcal{A}$ for every chain $\mathcal{C} \subset \mathcal{A}$. Then $\mathcal{A}$ contains a maximal element.

Zorn's Lemma now allows us to prove that every vector space has a basis. The proof does not help us find a concrete basis because Zorn's Lemma is an existence result rather than a constructive technique.

6.61 ***bases exist***

Every vector space has a basis.

Proof Suppose V is a vector space. If $\mathcal{C}$ is a chain of linearly independent subsets of V, then the union of all the sets in $\mathcal{C}$ is also a linearly independent subset of V (this holds because linear independence is a condition that is checked by considering finite subsets, and each finite subset of the union is contained in one of the elements of the chain).

Thus if $\mathcal{A}$ denotes the collection of linearly independent subsets of V, then $\mathcal{A}$ satisfies the hypothesis of Zorn's Lemma (6.60). Hence $\mathcal{A}$ contains a maximal element, which by 6.57 is a basis of V. ■

Now we can prove the promised result about the existence of discontinuous linear functionals on every infinite-dimensional normed vector space.

6.62 ***discontinuous linear functionals***

Every infinite-dimensional normed vector space has a discontinuous linear functional.

Proof Suppose V is an infinite-dimensional vector space. By 6.61, V has a basis $\{e_k\}_{k\in\Gamma}$. Because V is infinite-dimensional, Γ is not a finite set. Thus we can assume $\mathbf{Z}^+ \subset \Gamma$ (by relabeling a countable subset of Γ).

Define a linear functional $\varphi\colon V \to \mathbf{F}$ by setting $\varphi(e_j)$ equal to $j\|e_j\|$ for $j \in \mathbf{Z}^+$, setting $\varphi(e_j)$ equal to 0 for $j \in \Gamma \setminus \mathbf{Z}^+$, and extending linearly. More precisely, define a linear functional $\varphi\colon V \to \mathbf{F}$ by

$$\varphi\Big(\sum_{j\in\Omega} \alpha_j e_j\Big) = \sum_{j\in\Omega\cap\mathbf{Z}^+} \alpha_j j\|e_j\|$$

for every finite subset $\Omega \subset \Gamma$ and every family $\{\alpha_j\}_{j\in\Omega}$ in $\mathbf{F}$.

Because $\varphi(e_j) = j\|e_j\|$ for each $j \in \mathbf{Z}^+$, the linear functional φ is unbounded, completing the proof. ■

Hahn–Banach Theorem

In the last subsection, we showed that there exists a discontinuous linear functional on each infinite-dimensional normed vector space. Now we turn our attention to the existence of continuous linear functionals.

The existence of a nonzero continuous linear functional on each Banach space is not obvious. For example, consider the Banach space ℓ^∞/c_0, where ℓ^∞ is the Banach space of bounded sequences in $\mathbf{F}$ with

$$\|(a_1, a_2, \ldots)\|_\infty = \sup_{k\in\mathbf{Z}^+} |a_k|$$

and c_0 is the subspace of ℓ^∞ consisting of those sequences in $\mathbf{F}$ that have limit 0. The quotient space ℓ^∞/c_0 is an infinite-dimensional Banach space (see Exercise 15 in Section 6C). However, no one has ever exhibited a concrete nonzero linear functional on the Banach space ℓ^∞/c_0.

In this subsection, we show that infinite-dimensional normed vector spaces have plenty of continuous linear functionals. We do this by showing that a bounded linear functional on a subspace of a normed vector space can be extended to a bounded linear functional on the whole space without increasing its norm—this result is called the Hahn–Banach Theorem (6.69).

Completeness plays no role in this topic. Thus this subsection deals with normed vector spaces instead of Banach spaces.

We do most of the work needed to prove the Hahn–Banach Theorem in the next lemma, which shows that we can extend a linear functional to a subspace generated by one additional element, without increasing the norm. This one-element-at-a-time approach, when combined with a maximal object produced by Zorn's Lemma, gives us the desired extension to the full normed vector space.

If V is a real vector space, U is a subspace of V, and $h \in V$, then $U + \mathbf{R}h$ is the subspace of V defined by

$$U + \mathbf{R}h = \{f + \alpha h : f \in U \text{ and } \alpha \in \mathbf{R}\}.$$

6.63 *Extension Lemma*

Suppose V is a real normed vector space, U is a subspace of V, and $\psi\colon U \to \mathbf{R}$ is a bounded linear functional. Suppose $h \in V \setminus U$. Then ψ can be extended to a bounded linear functional $\varphi\colon U + \mathbf{R}h \to \mathbf{R}$ such that $\|\varphi\| = \|\psi\|$.

Proof Suppose $c \in \mathbf{R}$. Define $\varphi(h)$ to be c, and then extend φ linearly to $U + \mathbf{R}h$. Specifically, define $\varphi\colon U + \mathbf{R}h \to \mathbf{R}$ by

$$\varphi(f + \alpha h) = \psi(f) + \alpha c$$

for $f \in U$ and $\alpha \in \mathbf{R}$. Then φ is a linear functional on $U + \mathbf{R}h$.

Clearly $\varphi|_U = \psi$. Thus $\|\varphi\| \geq \|\psi\|$. We need to show that for some choice of $c \in \mathbf{R}$, the linear functional φ defined above satisfies the equation $\|\varphi\| = \|\psi\|$. In other words, we want

6.64 $$|\psi(f) + \alpha c| \leq \|\psi\|\,\|f + \alpha h\| \quad \text{for all } f \in U \text{ and all } \alpha \in \mathbf{R}.$$

It would be enough to have

6.65 $$|\psi(f) + c| \leq \|\psi\|\,\|f + h\| \quad \text{for all } f \in U,$$

because replacing f by $\frac{f}{\alpha}$ in the last inequality and then multiplying both sides by $|\alpha|$ would give 6.64.

Rewriting 6.65, we want to show that there exists $c \in \mathbf{R}$ such that

$$-\|\psi\|\,\|f + h\| \leq \psi(f) + c \leq \|\psi\|\,\|f + h\| \quad \text{for all } f \in U.$$

Equivalently, we want to show that there exists $c \in \mathbf{R}$ such that

$$-\|\psi\|\,\|f + h\| - \psi(f) \leq c \leq \|\psi\|\,\|f + h\| - \psi(f) \quad \text{for all } f \in U.$$

The existence of $c \in \mathbf{R}$ satisfying the line above follows from the inequality

6.66 $$\sup_{f \in U}\bigl(-\|\psi\|\,\|f + h\| - \psi(f)\bigr) \leq \inf_{g \in U}\bigl(\|\psi\|\,\|g + h\| - \psi(g)\bigr).$$

To prove the inequality above, suppose $f, g \in U$. Then

$$\begin{aligned} -\|\psi\|\,\|f + h\| - \psi(f) &\leq \|\psi\|(\|g + h\| - \|g - f\|) - \psi(f) \\ &= \|\psi\|(\|g + h\| - \|g - f\|) + \psi(g - f) - \psi(g) \\ &\leq \|\psi\|\,\|g + h\| - \psi(g). \end{aligned}$$

The inequality above proves 6.66, which completes the proof. ∎

Because our simplified form of Zorn's Lemma deals with set inclusions rather than more general orderings, we need to use the notion of the graph of a function.

6.67 Definition ***graph***

Suppose $T\colon V \to W$ is a function from a set V to a set W. Then the *graph* of T is denoted $\operatorname{graph}(T)$ and is the subset of $V \times W$ defined by

$$\operatorname{graph}(T) = \{(f, T(f)) \in V \times W : f \in V\}.$$

Formally, a function from a set V to a set W equals its graph as defined above. However, because we usually think of a function more intuitively as a mapping, the separate notion of the graph of a function remains useful.

The easy proof of the next result is left to the reader. The first bullet point below uses the vector space structure of $V \times W$, which is a vector space with natural operations of addition and scalar multiplication, as given in Exercise 10 in Section 6B.

6.68 ***function properties in terms of graphs***

Suppose V and W are normed vector spaces and $T\colon V \to W$ is a function.

(a) T is a linear map if and only if $\operatorname{graph}(T)$ is a subspace of $V \times W$.

(b) Suppose $U \subset V$ and $S\colon U \to W$ is a function. Then T is an extension of S if and only if $\operatorname{graph}(S) \subset \operatorname{graph}(T)$.

(c) If $T\colon V \to W$ is a linear map and $c \in [0, \infty)$, then $\|T\| \le c$ if and only if $\|g\| \le c\|f\|$ for all $(f, g) \in \operatorname{graph}(T)$.

The proof of the Extension Lemma (6.63) used inequalities that do not make sense when $\mathbf{F} = \mathbf{C}$. Thus the proof of the Hahn–Banach Theorem below requires some extra steps when $\mathbf{F} = \mathbf{C}$.

Hans Hahn (1879–1934) was a student and later a faculty member at the University of Vienna, where one of his PhD students was Kurt Gödel (1906–1978).

6.69 ***Hahn–Banach Theorem***

Suppose V is a normed vector space, U is a subspace of V, and $\psi\colon U \to \mathbf{F}$ is a bounded linear functional. Then ψ can be extended to a bounded linear functional on V whose norm equals $\|\psi\|$.

Proof First we consider the case where $\mathbf{F} = \mathbf{R}$. Let $\mathcal{A}$ be the collection of subsets E of $V \times \mathbf{R}$ that satisfy all the following conditions:

- $E = \operatorname{graph}(\varphi)$ for some linear functional φ on some subspace of V;
- $\operatorname{graph}(\psi) \subset E$;
- $|\alpha| \le \|\psi\|\,\|f\|$ for every $(f, \alpha) \in E$.

Then $\mathcal{A}$ satisfies the hypothesis of Zorn's Lemma (6.60). Thus $\mathcal{A}$ has a maximal element. The Extension Lemma (6.63) implies that this maximal element is the graph of a linear functional defined on all of V. This linear functional is an extension of ψ to V and it has norm $\|\psi\|$, completing the proof in the case where $\mathbf{F} = \mathbf{R}$.

Now consider the case where $\mathbf{F} = \mathbf{C}$. Define $\psi_1 \colon U \to \mathbf{R}$ by

$$\psi_1(f) = \operatorname{Re} \psi(f)$$

for $f \in U$. Then ψ_1 is an $\mathbf{R}$-linear map from U to $\mathbf{R}$ and $\|\psi_1\| \leq \|\psi\|$ (actually $\|\psi_1\| = \|\psi\|$, but we need only the inequality). Also,

$$\begin{aligned} \psi(f) &= \operatorname{Re} \psi(f) + i \operatorname{Im} \psi(f) \\ &= \psi_1(f) + i \operatorname{Im}\bigl(-i\psi(if)\bigr) \\ &= \psi_1(f) - i \operatorname{Re}\bigl(\psi(if)\bigr) \\ &= \psi_1(f) - i\psi_1(if) \qquad \text{6.70} \end{aligned}$$

for all $f \in U$.

Temporarily forget that complex scalar multiplication makes sense on V and temporarily think of V as a real normed vector space. The case of the result that we have already proved then implies that there exists an extension φ_1 of ψ_1 to an $\mathbf{R}$-linear functional $\varphi_1 \colon V \to \mathbf{R}$ with $\|\varphi_1\| = \|\psi_1\| \leq \|\psi\|$.

Motivated by 6.70, we define $\varphi \colon V \to \mathbf{C}$ by

$$\varphi(f) = \varphi_1(f) - i\varphi_1(if)$$

for $f \in V$. The equation above and 6.70 imply that φ is an extension of ψ to V. The equation above also implies that $\varphi(f+g) = \varphi(f) + \varphi(g)$ and $\varphi(\alpha f) = \alpha\varphi(f)$ for all $f, g \in V$ and all $\alpha \in \mathbf{R}$. Also,

$$\varphi(if) = \varphi_1(if) - i\varphi_1(-f) = \varphi_1(if) + i\varphi_1(f) = i\bigl(\varphi_1(f) - i\varphi_1(if)\bigr) = i\varphi(f).$$

The reader should use the equation above to show that φ is a $\mathbf{C}$-linear map.

The only part of the proof that remains is to show that $\|\varphi\| \leq \|\psi\|$. To do this, note that

$$|\varphi(f)|^2 = \varphi\bigl(\overline{\varphi(f)}f\bigr) = \varphi_1\bigl(\overline{\varphi(f)}f\bigr) \leq \|\psi\| \, \|\overline{\varphi(f)}f\| = \|\psi\| \, |\varphi(f)| \, \|f\|$$

for all $f \in V$, where the second equality holds because $\varphi\bigl(\overline{\varphi(f)}f\bigr) \in \mathbf{R}$. Dividing by $|\varphi(f)|$, we see from the line above that $|\varphi(f)| \leq \|\psi\| \, \|f\|$ for all $f \in V$ (no division necessary if $\varphi(f) = 0$). This implies that $\|\varphi\| \leq \|\psi\|$, completing the proof. ∎

We have given the special name *linear functionals* to linear maps into the scalar field $\mathbf{F}$. The vector space of bounded linear functionals now also gets a special name and a special notation.

6.71 Definition ***dual space***

Suppose V is a normed vector space. Then the *dual space* of V, denoted V', is the normed vector space consisting of the bounded linear functionals on V. In other words, $V' = \mathcal{B}(V, \mathbf{F})$.

By 6.47, the dual space of every normed vector space is a Banach space.

6.72 $\|f\| = \max\{|\varphi(f)| : \varphi \in V' \text{ and } \|\varphi\| = 1\}$

Suppose V is a normed vector space and $f \in V \setminus \{0\}$. Then there exists $\varphi \in V'$ such that $\|\varphi\| = 1$ and $\|f\| = \varphi(f)$.

Proof Let U be the 1-dimensional subspace of V defined by

$$U = \{\alpha f : \alpha \in \mathbf{F}\}.$$

Define $\psi\colon U \to \mathbf{F}$ by

$$\psi(\alpha f) = \alpha\|f\|$$

for $\alpha \in \mathbf{F}$. Then ψ is a linear functional on U with $\|\psi\| = 1$ and $\psi(f) = \|f\|$. The Hahn–Banach Theorem (6.69) implies that there exists an extension of ψ to a linear functional φ on V with $\|\varphi\| = 1$, completing the proof. ∎

The next result gives another beautiful application of the Hahn–Banach Theorem, with a useful necessary and sufficient condition for an element of a normed vector space to be in the closure of a subspace.

6.73 ***condition to be in the closure of a subspace***

Suppose U is a subspace of a normed vector space V and $h \in V$. Then $h \in \overline{U}$ if and only if $\varphi(h) = 0$ for every $\varphi \in V'$ such that $\varphi|_U = 0$.

Proof First suppose $h \in \overline{U}$. If $\varphi \in V'$ and $\varphi|_U = 0$, then $\varphi(h) = 0$ by the continuity of φ, completing the proof in one direction.

To prove the other direction, suppose now that $h \notin \overline{U}$. Define $\psi\colon U + \mathbf{F}h \to \mathbf{F}$ by

$$\psi(f + \alpha h) = \alpha$$

for $f \in U$ and $\alpha \in \mathbf{F}$. Then ψ is a linear functional on $U + \mathbf{F}h$ with $\operatorname{null}\psi = U$ and $\psi(h) = 1$.

Because $h \notin \overline{U}$, the closure of the null space of ψ does not equal $U + \mathbf{F}h$. Thus 6.52 implies that ψ is a bounded linear functional on $U + \mathbf{F}h$.

The Hahn–Banach Theorem (6.69) implies that ψ can be extended to a bounded linear functional φ on V. Thus we have found $\varphi \in V'$ such that $\varphi|_U = 0$ but $\varphi(h) \neq 0$, completing the proof in the other direction. ∎

EXERCISES 6D

1 Suppose V is a normed vector space and φ is a linear functional on V. Suppose $\alpha \in \mathbf{F} \setminus \{0\}$. Prove that the following are equivalent:

(a) φ is a bounded linear functional.

(b) $\varphi^{-1}(\alpha)$ is a closed subset of V.

(c) $\overline{\varphi^{-1}(\alpha)} \neq V$.

2 Suppose φ is a linear functional on a vector space V. Prove that if U is a subspace of V such that $\operatorname{null}\varphi \subset U$, then $U = \operatorname{null}\varphi$ or $U = V$.

3 Suppose φ and ψ are linear functionals on the same vector space. Prove that

$$\operatorname{null}\varphi \subset \operatorname{null}\psi$$

if and only if there exists $\alpha \in \mathbf{F}$ such that $\psi = \alpha\varphi$.

For the next two exercises, $\mathbf{F}^n$ ***should be endowed with the norm*** $\|\cdot\|_\infty$ ***as defined in Example 6.34.***

4 Suppose $n \in \mathbf{Z}^+$ and V is a normed vector space. Prove that every linear map from $\mathbf{F}^n$ to V is continuous.

5 Suppose $n \in \mathbf{Z}^+$, V is a normed vector space, and $T\colon \mathbf{F}^n \to V$ is a linear map that is one-to-one and onto V.

(a) Show that

$$\inf\{\|Tx\| : x \in \mathbf{F}^n \text{ and } \|x\|_\infty = 1\} > 0.$$

(b) Prove that $T^{-1}\colon V \to \mathbf{F}^n$ is a bounded linear map.

6 Suppose $n \in \mathbf{Z}^+$.

(a) Prove that all norms on $\mathbf{F}^n$ have the same convergent sequences, the same open sets, and the same closed sets.

(b) Prove that all norms on $\mathbf{F}^n$ make $\mathbf{F}^n$ into a Banach space.

7 Suppose V and W are normed vector spaces and V is finite-dimensional. Prove that every linear map from V to W is continuous.

8 Prove that every finite-dimensional normed vector space is a Banach space.

9 Prove that every finite-dimensional subspace of each normed vector space is closed.

10 Give a concrete example of an infinite-dimensional normed vector space and a basis of that normed vector space.

11 Show that the collection $\mathcal{A} = \{k\mathbf{Z} : k = 2, 3, 4, \ldots\}$ of subsets of $\mathbf{Z}$ satisfies the hypothesis of Zorn's Lemma (6.60).

12 Prove that every linearly independent family in a vector space can be extended to a basis of the vector space.

13 Suppose V is a normed vector space, U is a subspace of V, and $\psi\colon U \to \mathbf{R}$ is a bounded linear functional. Prove that ψ has a unique extension to a bounded linear functional φ on V with $\|\varphi\| = \|\psi\|$ if and only if

$$\sup_{f\in U}\bigl(-\|\psi\|\,\|f+h\| - \psi(f)\bigr) = \inf_{g\in U}\bigl(\|\psi\|\,\|g+h\| - \psi(g)\bigr)$$

for every $h \in V \setminus U$.

14 Show that there exists a linear functional $\varphi\colon \ell^\infty \to \mathbf{F}$ such that

$$|\varphi(a_1, a_2, \dots)| \leq \|(a_1, a_2, \dots)\|_\infty$$

for all $(a_1, a_2, \dots) \in \ell^\infty$ and

$$\varphi(a_1, a_2, \dots) = \lim_{k\to\infty} a_k$$

for all $(a_1, a_2, \dots) \in \ell^\infty$ such that the limit above on the right exists.

15 Suppose B is an open ball in a normed vector space V such that $0 \notin B$. Prove that there exists $\varphi \in V'$ such that

$$\operatorname{Re} \varphi(f) > 0$$

for all $f \in B$.

16 Show that the dual space of each infinite-dimensional normed vector space is infinite-dimensional.

A normed vector space is called* separable *if it has a countable subset whose closure equals the whole space.

17 Suppose V is a separable normed vector space. Explain how the Hahn–Banach Theorem (6.69) for V can be proved without using any results (such as Zorn's Lemma) that depend upon the Axiom of Choice.

18 Suppose V is a normed vector space such that the dual space V' is a separable Banach space. Prove that V is separable.

19 Prove that the dual of the Banach space $C([0,1])$ is not separable; here the norm on $C([0,1])$ is defined by $\|f\| = \sup_{[0,1]} |f|$.

The* double dual space *of a normed vector space is defined to be the dual space of the dual space. If V is a normed vector space, then the double dual space of V is denoted by V''; thus $V'' = (V')'$. The norm on V'' is defined to be the norm it receives as the dual space of V'.

20 Define $\Phi\colon V \to V''$ by

$$(\Phi f)(\varphi) = \varphi(f)$$

for $f \in V$ and $\varphi \in V'$. Show that $\|\Phi f\| = \|f\|$ for every $f \in V$.
[*The map Φ defined above is called the* canonical isometry *of V into V''.*]

21 Suppose V is an infinite-dimensional normed vector space. Show that there is a convex subset U of V such that $\overline{U} = V$ and such that the complement $V \setminus U$ is also a convex subset of V with $\overline{V \setminus U} = V$.
[*See 8.25 for the definition of a convex set. This exercise should stretch your geometric intuition because this behavior cannot happen in finite dimensions.*]

6E Consequences of Baire's Theorem

This section focuses on several important results about Banach spaces that depend upon Baire's Theorem. This result was first proved by René-Louis Baire (1874–1932) as part of his 1899 doctoral dissertation at École Normale Supérieure (Paris).

Even though our interest lies primarily in applications to Banach spaces, the proper setting for Baire's Theorem is the more general context of complete metric spaces.

The result here called Baire's Theorem *is often called the* Baire Category Theorem. *This book uses the shorter name of this result because we do not need the categories introduced by Baire. Furthermore, the use of the word* category *in this context can be confusing because Baire's categories have no connection with the category theory that developed decades after Baire's work.*

Baire's Theorem

We begin with some key topological notions.

6.74 Definition *interior*

Suppose U is a subset of a metric space V. The *interior* of U, denoted $\operatorname{int} U$, is the set of $f \in U$ such that some open ball of V centered at f with positive radius is contained in U.

You should verify the following elementary facts about the interior.

- The interior of each subset of a metric space is open.
- The interior of a subset U of a metric space V is the largest open subset of V contained in U.

6.75 Definition *dense*

A subset U of a metric space V is called *dense* in V if $\overline{U} = V$.

For example, $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$ are both dense in $\mathbf{R}$, where $\mathbf{R}$ has its standard metric $d(x, y) = |x - y|$.

You should verify the following elementary facts about dense subsets.

- A subset U of a metric space V is dense in V if and only if every nonempty open subset of V contains at least one element of U.
- A subset U of a metric space V has an empty interior if and only if $V \setminus U$ is dense in V.

The proof of the next result uses the following fact, which you should first prove: If G is an open subset of a metric space V and $f \in G$, then there exists $r > 0$ such that $\overline{B}(f, r) \subset G$.

6.76 *Baire's Theorem*

(a) A complete metric space is not the countable union of closed subsets with empty interior.

(b) The countable intersection of dense open subsets of a complete metric space is nonempty.

Proof We will prove (b) and then use (b) to prove (a).

To prove (b), suppose (V, d) is a complete metric space and $G_1, G_2, \dots$ is a sequence of dense open subsets of V. We need to show that $\bigcap_{k=1}^{\infty} G_k \neq \varnothing$.

Let $f_1 \in G_1$ and let $r_1 \in (0, 1)$ be such that $\overline{B}(f_1, r_1) \subset G_1$. Now suppose $n \in \mathbf{Z}^+$, and $f_1, \dots, f_n$ and $r_1, \dots, r_n$ have been chosen such that

6.77 $$\overline{B}(f_1, r_1) \supset \overline{B}(f_2, r_2) \supset \cdots \supset \overline{B}(f_n, r_n)$$

and

6.78 $$r_j \in \left(0, \tfrac{1}{j}\right) \quad \text{and} \quad \overline{B}(f_j, r_j) \subset G_j \quad \text{for } j = 1, \dots, n.$$

Because $B(f_n, r_n)$ is an open subset of V and G_{n+1} is dense in V, there exists $f_{n+1} \in B(f_n, r_n) \cap G_{n+1}$. Let $r_{n+1} \in \left(0, \frac{1}{n+1}\right)$ be such that

$$\overline{B}(f_{n+1}, r_{n+1}) \subset \overline{B}(f_n, r_n) \cap G_{n+1}.$$

Thus we inductively construct a sequence $f_1, f_2, \dots$ that satisfies 6.77 and 6.78 for all $n \in \mathbf{Z}^+$.

If $j \in \mathbf{Z}^+$, then 6.77 and 6.78 imply that

6.79 $$f_k \in \overline{B}(f_j, r_j) \quad \text{and} \quad d(f_j, f_k) \leq r_j < \tfrac{1}{j} \quad \text{for all } k > j.$$

Hence $f_1, f_2, \dots$ is a Cauchy sequence. Because (V, d) is a complete metric space, there exists $f \in V$ such that $\lim_{k \to \infty} f_k = f$.

Now 6.79 and 6.78 imply that for each $j \in \mathbf{Z}^+$, we have $f \in \overline{B}(f_j, r_j) \subset G_j$. Hence $f \in \bigcap_{k=1}^{\infty} G_k$, which means that $\bigcap_{k=1}^{\infty} G_k$ is not the empty set, completing the proof of (b).

To prove (a), suppose (V, d) is a complete metric space and $F_1, F_2, \dots$ is a sequence of closed subsets of V with empty interior. Then $V \setminus F_1, V \setminus F_2, \dots$ is a sequence of dense open subsets of V. Now (b) implies that

$$\varnothing \neq \bigcap_{k=1}^{\infty} (V \setminus F_k).$$

Taking complements of both sides above, we conclude that

$$V \neq \bigcup_{k=1}^{\infty} F_k,$$

completing the proof of (a). ■

Because

$$\mathbf{R} = \bigcup_{x \in \mathbf{R}} \{x\}$$

and each set $\{x\}$ has empty interior in $\mathbf{R}$, Baire's Theorem implies $\mathbf{R}$ is uncountable. Thus we have yet another proof that $\mathbf{R}$ is uncountable, different than Cantor's original diagonal proof and different from the proof via measure theory (see 2.17).

The next result is another nice consequence of Baire's Theorem.

6.80 ***the set of irrational numbers is not a countable union of closed sets***

There does not exist a countable collection of closed subsets of $\mathbf{R}$ whose union equals $\mathbf{R} \setminus \mathbf{Q}$.

Proof This will be a proof by contradiction. Suppose $F_1, F_2, \ldots$ is a countable collection of closed subsets of $\mathbf{R}$ whose union equals $\mathbf{R} \setminus \mathbf{Q}$. Thus each F_k contains no rational numbers, which implies that each F_k has empty interior. Now

$$\mathbf{R} = \Bigl(\bigcup_{r \in \mathbf{Q}} \{r\}\Bigr) \cup \Bigl(\bigcup_{k=1}^{\infty} F_k\Bigr).$$

The equation above writes the complete metric space $\mathbf{R}$ as a countable union of closed sets with empty interior, which contradicts Baire's Theorem [6.76(a)]. This contradiction completes the proof. ■

Open Mapping Theorem and Inverse Mapping Theorem

The next result shows that a surjective bounded linear map from one Banach space onto another Banach space maps open sets to open sets. As shown in Exercises 10 and 11, this result can fail if the hypothesis that both spaces are Banach spaces is weakened to allow either of the spaces to be a normed vector space.

6.81 ***Open Mapping Theorem***

Suppose V and W are Banach spaces and T is a bounded linear map of V onto W. Then $T(G)$ is an open subset of W for every open subset G of V.

Proof Let B denote the open unit ball $B(0,1) = \{f \in V : \|f\| < 1\}$ of V. For any open ball $B(f,a)$ in V, the linearity of T implies that

$$T\bigl(B(f,a)\bigr) = Tf + aT(B).$$

Suppose G is an open subset of V. If $f \in G$, then there exists $a > 0$ such that $B(f,a) \subset G$. If we can show that $0 \in \operatorname{int} T(B)$, then the equation above shows that $Tf \in \operatorname{int} T\bigl(B(f,a)\bigr)$. This would imply that $T(G)$ is an open subset of W. Thus to complete the proof we need only show that $T(B)$ contains some open ball centered at 0.

The surjectivity and linearity of T imply that

$$W = \bigcup_{k=1}^{\infty} T(kB) = \bigcup_{k=1}^{\infty} kT(B).$$

Thus $W = \bigcup_{k=1}^{\infty} \overline{kT(B)}$. Baire's Theorem [6.76(a)] now implies that $\overline{kT(B)}$ has a nonempty interior for some $k \in \mathbf{Z}^+$. The linearity of T allows us to conclude that $\overline{T(B)}$ has a nonempty interior.

Thus there exists $g \in B$ such that $Tg \in \operatorname{int} \overline{T(B)}$. Hence

$$0 \in \operatorname{int} \overline{T(B - g)} \subset \operatorname{int} \overline{T(2B)} = \operatorname{int} \overline{2T(B)}.$$

Thus there exists $r > 0$ such that $\overline{B}(0, 2r) \subset \overline{2T(B)}$ [here $\overline{B}(0, 2r)$ is the closed ball in W centered at 0 with radius $2r$]. Hence $\overline{B}(0, r) \subset \overline{T(B)}$. The definition of what it means to be in the closure of $T(B)$ [see 6.7] now shows that

$$h \in W \text{ and } \|h\| \leq r \text{ and } \varepsilon > 0 \implies \exists f \in B \text{ such that } \|h - Tf\| < \varepsilon.$$

For arbitrary $h \neq 0$ in W, applying the result in the line above to $\frac{r}{\|h\|}h$ shows that

6.82 $$h \in W \text{ and } \varepsilon > 0 \implies \exists f \in \frac{\|h\|}{r} B \text{ such that } \|h - Tf\| < \varepsilon.$$

Now suppose $g \in W$ and $\|g\| < 1$. Applying 6.82 with $h = g$ and $\varepsilon = \frac{1}{2}$, we see that

$$\text{there exists } f_1 \in \tfrac{1}{r}B \text{ such that } \|g - Tf_1\| < \tfrac{1}{2}.$$

Now applying 6.82 with $h = g - Tf_1$ and $\varepsilon = \frac{1}{4}$, we see that

$$\text{there exists } f_2 \in \tfrac{1}{2r}B \text{ such that } \|g - Tf_1 - Tf_2\| < \tfrac{1}{4}.$$

Applying 6.82 again, this time with $h = g - Tf_1 - Tf_2$ and $\varepsilon = \frac{1}{8}$, we see that

$$\text{there exists } f_3 \in \tfrac{1}{4r}B \text{ such that } \|g - Tf_1 - Tf_2 - Tf_3\| < \tfrac{1}{8}.$$

Continue in this pattern, constructing a sequence $f_1, f_2, \ldots$ in V. Let

$$f = \sum_{k=1}^{\infty} f_k,$$

where the infinite sum converges in V because

$$\sum_{k=1}^{\infty} \|f_k\| < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}r} = \frac{2}{r};$$

here we are using 6.41 (this is the place in the proof where we use the hypothesis that V is a Banach space). The inequality displayed above shows that $\|f\| < \frac{2}{r}$.

Because

$$\|g - Tf_1 - Tf_2 - \cdots - Tf_n\| < \frac{1}{2^n}$$

and because T is a continuous linear map, we have $g = Tf$.

We have now shown that $B(0, 1) \subset \frac{2}{r}T(B)$. Thus $\frac{r}{2}B(0, 1) \subset T(B)$, completing the proof. ∎

The next result provides the useful information that if a bounded linear map from one Banach space to another Banach space has an algebraic inverse (meaning that the linear map is injective and surjective), then the inverse mapping is automatically bounded.

The Open Mapping Theorem was first proved by Banach and his colleague Juliusz Schauder (1899–1943) in 1929–1930.

6.83 *Bounded Inverse Theorem*

Suppose V and W are Banach spaces and T is a one-to-one bounded linear map from V onto W. Then T^{-1} is a bounded linear map from W onto V.

Proof The verification that T^{-1} is a linear map from W to V is left to the reader.

To prove that T^{-1} is bounded, suppose G is an open subset of V. Then

$$(T^{-1})^{-1}(G) = T(G).$$

By the Open Mapping Theorem (6.81), $T(G)$ is an open subset of W. Thus the equation above shows that the inverse image under the function T^{-1} of every open set is open. By the equivalence of parts (a) and (c) of 6.11, this implies that T^{-1} is continuous. Thus T^{-1} is a bounded linear map (by 6.48). ■

The result above shows that completeness for normed vector spaces sometimes plays a role analogous to compactness for metric spaces (think of the theorem stating that a continuous one-to-one function from a compact metric space onto another compact metric space has an inverse that is also continuous).

Closed Graph Theorem

Suppose V and W are normed vector spaces. Then $V \times W$ is a vector space with the natural operations of addition and scalar multiplication as defined in Exercise 10 in Section 6B. There are several natural norms on $V \times W$ that make $V \times W$ into a normed vector space; the choice used in the next result seems to be the easiest. The proof of the next result is left to the reader as an exercise.

6.84 *product of Banach spaces*

Suppose V and W are Banach spaces. Then $V \times W$ is a Banach space if given the norm defined by

$$\|(f, g)\| = \max\{\|f\|, \|g\|\}$$

for $f \in V$ and $g \in W$. With this norm, a sequence $(f_1, g_1), (f_2, g_2), \dots$ in $V \times W$ converges to (f, g) if and only if $\lim_{k\to\infty} f_k = f$ and $\lim_{k\to\infty} g_k = g$.

The next result gives a terrific way to show that a linear map between Banach spaces is bounded. The proof is remarkably clean because the hard work has been done in the proof of the Open Mapping Theorem (which was used to prove the Bounded Inverse Theorem).

6.85 *Closed Graph Theorem*

Suppose V and W are Banach spaces and T is a function from V to W. Then T is a bounded linear map if and only if $\operatorname{graph}(T)$ is a closed subspace of $V \times W$.

Proof First suppose T is a bounded linear map. Suppose $(f_1, Tf_1), (f_2, Tf_2), \ldots$ is a sequence in $\operatorname{graph}(T)$ converging to $(f, g) \in V \times W$. Thus

$$\lim_{k\to\infty} f_k = f \quad \text{and} \quad \lim_{k\to\infty} Tf_k = g.$$

Because T is continuous, the first equation above implies that $\lim_{k\to\infty} Tf_k = Tf$; when combined with the second equation above this implies that $g = Tf$. Thus $(f, g) = (f, Tf) \in \operatorname{graph}(T)$, which implies that $\operatorname{graph}(T)$ is closed, completing the proof in one direction.

To prove the other direction, now suppose $\operatorname{graph}(T)$ is a closed subspace of $V \times W$. Thus $\operatorname{graph}(T)$ is a Banach space with the norm that it inherits from $V \times W$ [from 6.84 and 6.16(b)]. Consider the linear map $S\colon \operatorname{graph}(T) \to V$ defined by

$$S(f, Tf) = f.$$

Then

$$\|S(f, Tf)\| = \|f\| \leq \max\{\|f\|, \|Tf\|\} = \|(f, Tf)\|$$

for all $f \in V$. Thus S is a bounded linear map from $\operatorname{graph}(T)$ onto V with $\|S\| \leq 1$. Clearly S is injective. Thus the Bounded Inverse Theorem (6.83) implies that S^{-1} is bounded. Because $S^{-1}\colon V \to \operatorname{graph}(T)$ satisfies the equation $S^{-1}f = (f, Tf)$, we have

$$\begin{aligned}
\|Tf\| &\leq \max\{\|f\|, \|Tf\|\} \\
&= \|(f, Tf)\| \\
&= \|S^{-1}f\| \\
&\leq \|S^{-1}\|\,\|f\|
\end{aligned}$$

for all $f \in V$. The inequality above implies that T is a bounded linear map with $\|T\| \leq \|S^{-1}\|$, completing the proof. ■

Principle of Uniform Boundedness

The next result states that a family of bounded linear maps on a Banach space that is pointwise bounded is bounded in norm (which means that it is uniformly bounded as a collection of maps on the unit ball). This result is sometimes called the Banach–Steinhaus Theorem. Exercise 17 is also sometimes called the Banach–Steinhaus Theorem.

The Principle of Uniform Boundedness was proved in 1927 by Banach and Hugo Steinhaus (1887–1972). Steinhaus recruited Banach to advanced mathematics after overhearing him discuss Lebesgue integration in a park.

6.86 *Principle of Uniform Boundedness*

Suppose V is a Banach space, W is a normed vector space, and $\mathcal{A}$ is a family of bounded linear maps from V to W such that

$$\sup\{\|Tf\| : T \in \mathcal{A}\} < \infty \text{ for every } f \in V.$$

Then

$$\sup\{\|T\| : T \in \mathcal{A}\} < \infty.$$

Proof Our hypothesis implies that

$$V = \bigcup_{n=1}^{\infty} \underbrace{\{f \in V : \|Tf\| \leq n \text{ for all } T \in \mathcal{A}\}}_{V_n},$$

where V_n is defined by the expression above. Because each $T \in \mathcal{A}$ is continuous, V_n is a closed subset of V for each $n \in \mathbf{Z}^+$. Thus Baire's Theorem [6.76(a)] and the equation above imply that there exist $n \in \mathbf{Z}^+$ and $h \in V$ and $r > 0$ such that

6.87 $$B(h, r) \subset V_n.$$

Now suppose $g \in V$ and $\|g\| < 1$. Thus $rg + h \in B(h, r)$. Hence if $T \in \mathcal{A}$, then 6.87 implies $\|T(rg + h)\| \leq n$, which implies that

$$\|Tg\| = \left\| \frac{T(rg + h)}{r} - \frac{Th}{r} \right\| \leq \frac{\|T(rg + h)\|}{r} + \frac{\|Th\|}{r} \leq \frac{n + \|Th\|}{r}.$$

Thus

$$\sup\{\|T\| : T \in \mathcal{A}\} \leq \frac{n + \sup\{\|Th\| : T \in \mathcal{A}\}}{r} < \infty,$$

completing the proof. ■

EXERCISES 6E

1 Suppose U is a subset of a metric space V. Show that U is dense in V if and only if every nonempty open subset of V contains at least one element of U.

2 Suppose U is a subset of a metric space V. Show that U has an empty interior if and only if $V \setminus U$ is dense in V.

3 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $(\operatorname{int} U) \cup (\operatorname{int} W) = \operatorname{int}(U \cup W)$.

4 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $(\operatorname{int} U) \cap (\operatorname{int} W) = \operatorname{int}(U \cap W)$.

5 Suppose

$$X = \{0\} \cup \bigcup_{k=1}^{\infty} \{\tfrac{1}{k}\}$$

and $d(x,y) = |x - y|$ for $x, y \in X$.

(a) Show that (X, d) is a complete metric space.

(b) Each set of the form $\{x\}$ for $x \in X$ is a closed subset of $\mathbf{R}$ that has an empty interior as a subset of $\mathbf{R}$. Clearly X is a countable union of such sets. Explain why this does not violate the statement of Baire's Theorem that a complete metric space is not the countable union of closed subsets with empty interior.

6 Give an example of a metric space that is the countable union of closed subsets with empty interior.
[*This exercise shows that the completeness hypothesis in Baire's Theorem cannot be dropped.*]

7 (a) Define $f\colon \mathbf{R} \to \mathbf{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational,} \\ \frac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive integer} \\ & \text{such that } a = \frac{m}{n} \text{ for some integer } m. \end{cases}$$

At which numbers in $\mathbf{R}$ is f continuous?

(b) Show that there does not exist a countable collection of open subsets of $\mathbf{R}$ whose intersection equals $\mathbf{Q}$.

(c) Show that there does not exist a function $f\colon \mathbf{R} \to \mathbf{R}$ such that f is continuous at each element of $\mathbf{Q}$ and discontinuous at each element of $\mathbf{R} \setminus \mathbf{Q}$.

8 Suppose (X, d) is a complete metric space and $G_1, G_2, \dots$ is a sequence of dense open subsets of X. Prove that $\bigcap_{k=1}^{\infty} G_k$ is a dense subset of X.

9 Prove that there does not exist an infinite-dimensional Banach space with a countable basis.
[*This exercise implies, for example, that there is not a norm that makes the vector space of polynomials with coefficients in* **F** *into a Banach space.*]

10 Give an example of a Banach space V, a normed vector space W, a bounded linear map T of V onto W, and an open subset G of V such that $T(G)$ is not an open subset of W.
[*This exercise shows that the hypothesis in the Open Mapping Theorem that W is a Banach space cannot be relaxed to the hypothesis that W is a normed vector space.*]

11 Show that there exists a normed vector space V, a Banach space W, a bounded linear map T of V onto W, and an open subset G of V such that $T(G)$ is not an open subset of W.
[*This exercise shows that the hypothesis in the Open Mapping Theorem that V is a Banach space cannot be relaxed to the hypothesis that V is a normed vector space.*]

A linear map $T\colon V \to W$ from a normed vector space V to a normed vector space W is called* bounded below *if there exists $c \in (0,\infty)$ such that $\|f\| \le c\|Tf\|$ for all $f \in V$.

12 Suppose $T\colon V \to W$ is a bounded linear map from a Banach space V to a Banach space W. Prove that T is bounded below if and only if T is injective and the range of T is a closed subspace of W.

13 Give an example of a Banach space V, a normed vector space W, and a one-to-one bounded linear map T of V onto W such that T^{-1} is not a bounded linear map of W onto V.
[*This exercise shows that the hypothesis in the Bounded Inverse Theorem* (*6.83*) *that W is a Banach space cannot be relaxed to the hypothesis that W is a normed vector space.*]

14 Show that there exists a normed space V, a Banach space W, and a one-to-one bounded linear map T of V onto W such that T^{-1} is not a bounded linear map of W onto V.
[*This exercise shows that the hypothesis in the Bounded Inverse Theorem* (*6.83*) *that V is a Banach space cannot be relaxed to the hypothesis that V is a normed vector space.*]

15 Prove 6.84.

16 Suppose V is a Banach space with norm $\|\cdot\|$ and that $\varphi\colon V \to \mathbf{F}$ is a linear functional. Define another norm $\|\cdot\|_\varphi$ on V by

$$\|f\|_\varphi = \|f\| + |\varphi(f)|.$$

Prove that if V is a Banach space with the norm $\|\cdot\|_\varphi$, then φ is a continuous linear functional on V (with the original norm).

17 Suppose V is a Banach space, W is a normed vector space, and $T_1, T_2, \ldots$ is a sequence of bounded linear maps from V to W such that $\lim_{k\to\infty} T_k f$ exists for each $f \in V$. Define $T\colon V \to W$ by

$$Tf = \lim_{k\to\infty} T_k f$$

for $f \in V$. Prove that T is a bounded linear map from V to W.
[*This result states that the pointwise limit of a sequence of bounded linear maps on a Banach space is a bounded linear map.*]

18 Suppose V is a normed vector space and B is a subset of V such that

$$\sup_{f\in B} |\varphi(f)| < \infty$$

for every $\varphi \in V'$. Prove that $\sup\limits_{f\in B} \|f\| < \infty$.

19 Suppose $T\colon V \to W$ is a linear map from a Banach space V to a Banach space W such that

$$\varphi \circ T \in V' \text{ for all } \varphi \in W'.$$

Prove that T is a bounded linear map.

Chapter 7
L^p Spaces

Fix a measure space $(X, \mathcal{S}, \mu)$ and a positive number p. We begin this chapter by looking at the vector space of measurable functions $f\colon X \to \mathbf{F}$ such that

$$\int |f|^p \, d\mu < \infty.$$

Important results called Hölder's inequality and Minkowski's inequality help us investigate this vector space. A useful class of Banach spaces appears when we identify functions that differ only on a set of measure 0 and require $p \geq 1$.



The main building of the Swiss Federal Institute of Technology (ETH Zürich). Hermann Minkowski (1864–1909) taught at this university from 1896 to 1902. During this time, Albert Einstein (1879–1955) was a student in several of Minkowski's mathematics classes. Minkowski later created mathematics that helped explain Einstein's special theory of relativity.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_7

7A $\mathcal{L}^p(\mu)$

Hölder's Inequality

Our next major goal is to define an important class of vector spaces that generalize the vector spaces $\mathcal{L}^1(\mu)$ and ℓ^1 introduced in the last two bullet points of Example 6.32. We begin this process with the definition below. The terminology *p-norm* introduced below is convenient, even though it is not necessarily a norm.

7.1 Definition $\|f\|_p$

Suppose that $(X, \mathcal{S}, \mu)$ is a measure space, $0 < p < \infty$, and $f\colon X \to \mathbf{F}$ is $\mathcal{S}$-measurable. Then the *p-norm* of f is denoted by $\|f\|_p$ and is defined by

$$\|f\|_p = \Big(\int |f|^p \, d\mu\Big)^{1/p}.$$

Also, $\|f\|_\infty$, which is called the *essential supremum* of f, is defined by

$$\|f\|_\infty = \inf\big\{t > 0 : \mu(\{x \in X : |f(x)| > t\}) = 0\big\}.$$

The exponent $1/p$ appears in the definition of the p-norm $\|f\|_p$ because we want the equation $\|\alpha f\|_p = |\alpha| \, \|f\|_p$ to hold for all $\alpha \in \mathbf{F}$.

For $0 < p < \infty$, the p-norm $\|f\|_p$ does not change if f changes on a set of μ-measure 0. By using the essential supremum rather than the supremum in the definition of $\|f\|_\infty$, we arrange for the ∞-norm $\|f\|_\infty$ to enjoy this same property. Think of $\|f\|_\infty$ as the smallest that you can make the supremum of $|f|$ after modifications on sets of measure 0.

7.2 Example *p-norm for counting measure*

Suppose μ is counting measure on $\mathbf{Z}^+$. If $a = (a_1, a_2, \dots)$ is a sequence in $\mathbf{F}$ and $0 < p < \infty$, then

$$\|a\|_p = \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{1/p} \quad \text{and} \quad \|a\|_\infty = \sup\{|a_k| : k \in \mathbf{Z}^+\}.$$

Note that for counting measure, the essential supremum and the supremum are the same because in this case there are no sets of measure 0 other than the empty set.

Now we can define our generalization of $\mathcal{L}^1(\mu)$, which was defined in the second-to-last bullet point of Example 6.32.

7.3 Definition $\mathcal{L}^p(\mu)$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 < p \le \infty$. The *Lebesgue space* $\mathcal{L}^p(\mu)$, sometimes denoted $\mathcal{L}^p(X, \mathcal{S}, \mu)$, is defined to be the set of $\mathcal{S}$-measurable functions $f\colon X \to \mathbf{F}$ such that $\|f\|_p < \infty$.

7.4 Example ℓ^p

When μ is counting measure on $\mathbf{Z}^+$, the set $\mathcal{L}^p(\mu)$ is often denoted by ℓ^p (pronounced *little el-p*). Thus if $0 < p < \infty$, then

$$\ell^p = \{(a_1, a_2, \dots) : \text{each } a_k \in \mathbf{F} \text{ and } \sum_{k=1}^{\infty} |a_k|^p < \infty\}$$

and

$$\ell^\infty = \{(a_1, a_2, \dots) : \text{each } a_k \in \mathbf{F} \text{ and } \sup_{k \in \mathbf{Z}^+} |a_k| < \infty\}.$$

Inequality 7.5(a) below provides an easy proof that $\mathcal{L}^p(\mu)$ is closed under addition. Soon we will prove Minkowski's inequality (7.14), which provides an important improvement of 7.5(a) when $p \geq 1$ but is more complicated to prove.

7.5 $\mathcal{L}^p(\mu)$ *is a vector space*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 < p < \infty$. Then

(a)
$$\|f + g\|_p^p \leq 2^p(\|f\|_p^p + \|g\|_p^p)$$

and

(b)
$$\|\alpha f\|_p = |\alpha| \, \|f\|_p$$

for all $f, g \in \mathcal{L}^p(\mu)$ and all $\alpha \in \mathbf{F}$. Furthermore, with the usual operations of addition and scalar multiplication of functions, $\mathcal{L}^p(\mu)$ is a vector space.

Proof Suppose $f, g \in \mathcal{L}^p(\mu)$. If $x \in X$, then

$$\begin{aligned} |f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p \\ &\leq (2\max\{|f(x)|, |g(x)|\})^p \\ &\leq 2^p(|f(x)|^p + |g(x)|^p). \end{aligned}$$

Integrating both sides of the inequality above with respect to μ gives the desired inequality

$$\|f + g\|_p^p \leq 2^p(\|f\|_p^p + \|g\|_p^p).$$

This inequality implies that if $\|f\|_p < \infty$ and $\|g\|_p < \infty$, then $\|f + g\|_p < \infty$. Thus $\mathcal{L}^p(\mu)$ is closed under addition.

The proof that

$$\|\alpha f\|_p = |\alpha| \, \|f\|_p$$

follows easily from the definition of $\|\cdot\|_p$. This equality implies that $\mathcal{L}^p(\mu)$ is closed under scalar multiplication.

Because $\mathcal{L}^p(\mu)$ contains the constant function 0 and is closed under addition and scalar multiplication, $\mathcal{L}^p(\mu)$ is a subspace of $\mathbf{F}^X$ and thus is a vector space. ∎

What we call the *dual exponent* in the definition below is often called the *conjugate exponent* or the *conjugate index*. However, the terminology *dual exponent* conveys more meaning because of results (7.25 and 7.26) that we will see in the next section.

7.6 Definition ***dual exponent; p'***

For $1 \le p \le \infty$, the *dual exponent* of p is denoted by p' and is the element of $[1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

7.7 Example *dual exponents*

$$1' = \infty, \quad \infty' = 1, \quad 2' = 2, \quad 4' = 4/3, \quad (4/3)' = 4$$

The result below is a key tool in proving Hölder's inequality (7.9).

7.8 ***Young's inequality***

Suppose $1 < p < \infty$. Then

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

for all $a \ge 0$ and $b \ge 0$.

William Henry Young (1863–1942) published what is now called Young's inequality in 1912.

Proof Fix $b > 0$ and define a function $f\colon (0, \infty) \to \mathbf{R}$ by

$$f(a) = \frac{a^p}{p} + \frac{b^{p'}}{p'} - ab.$$

Thus $f'(a) = a^{p-1} - b$. Hence f is decreasing on the interval $\big(0, b^{1/(p-1)}\big)$ and f is increasing on the interval $\big(b^{1/(p-1)}, \infty\big)$. Thus f has a global minimum at $b^{1/(p-1)}$. A tiny bit of arithmetic [use $p/(p-1) = p'$] shows that $f\big(b^{1/(p-1)}\big) = 0$. Thus $f(a) \ge 0$ for all $a \in (0, \infty)$, which implies the desired inequality. ∎

The important result below furnishes a key tool that is used in the proof of Minkowski's inequality (7.14).

7.9 ***Hölder's inequality***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $1 \le p \le \infty$, and $f, h\colon X \to \mathbf{F}$ are $\mathcal{S}$-measurable. Then

$$\|fh\|_1 \le \|f\|_p \, \|h\|_{p'}.$$

Proof Suppose $1 < p < \infty$, leaving the cases $p = 1$ and $p = \infty$ as exercises for the reader.

First consider the special case where $\|f\|_p = \|h\|_{p'} = 1$. Young's inequality (7.8) tells us that

$$|f(x)h(x)| \leq \frac{|f(x)|^p}{p} + \frac{|h(x)|^{p'}}{p'}$$

for all $x \in X$. Integrating both sides of the inequality above with respect to μ shows that $\|fh\|_1 \leq 1 = \|f\|_p \|h\|_{p'}$, completing the proof in this special case.

Hölder's inequality was proved in 1889 by Otto Hölder (1859–1937).

If $\|f\|_p = 0$ or $\|h\|_{p'} = 0$, then $\|fh\|_1 = 0$ and the desired inequality holds. Similarly, if $\|f\|_p = \infty$ or $\|h\|_{p'} = \infty$, then the desired inequality clearly holds. Thus we assume that $0 < \|f\|_p < \infty$ and $0 < \|h\|_{p'} < \infty$.

Now define $\mathcal{S}$-measurable functions $f_1, h_1 \colon X \to \mathbf{F}$ by

$$f_1 = \frac{f}{\|f\|_p} \quad \text{and} \quad h_1 = \frac{h}{\|h\|_{p'}}.$$

Then $\|f_1\|_p = 1$ and $\|h_1\|_{p'} = 1$. By the result for our special case, we have $\|f_1 h_1\|_1 \leq 1$, which implies that $\|fh\|_1 \leq \|f\|_p \|h\|_{p'}$. ■

The next result gives a key containment among Lebesgue spaces with respect to a finite measure. Note the crucial role that Hölder's inequality plays in the proof.

7.10 $\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu)$ ***if*** $p < q$ ***and*** $\mu(X) < \infty$

Suppose $(X, \mathcal{S}, \mu)$ is a finite measure space and $0 < p < q < \infty$. Then

$$\|f\|_p \leq \mu(X)^{(q-p)/(pq)} \|f\|_q$$

for all $f \in \mathcal{L}^q(\mu)$. Furthermore, $\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu)$.

Proof Fix $f \in \mathcal{L}^q(\mu)$. Let $r = \frac{q}{p}$. Thus $r > 1$. A short calculation shows that $r' = \frac{q}{q-p}$. Now Hölder's inequality (7.9) with p replaced by r and f replaced by $|f|^p$ and h replaced by the constant function 1 gives

$$\begin{aligned}\int |f|^p \, d\mu &\leq \Bigl(\int (|f|^p)^r \, d\mu\Bigr)^{1/r} \Bigl(\int 1^{r'} \, d\mu\Bigr)^{1/r'} \\ &= \mu(X)^{(q-p)/q} \Bigl(\int |f|^q \, d\mu\Bigr)^{p/q}.\end{aligned}$$

Now raise both sides of the inequality above to the power $\frac{1}{p}$, getting

$$\Bigl(\int |f|^p \, d\mu\Bigr)^{1/p} \leq \mu(X)^{(q-p)/(pq)} \Bigl(\int |f|^q \, d\mu\Bigr)^{1/q},$$

which is the desired inequality.

The inequality above shows that $f \in \mathcal{L}^p(\mu)$. Thus $\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu)$. ■

7.11 Example $\mathcal{L}^p(E)$

We adopt the common convention that if E is a Borel (or Lebesgue measurable) subset of $\mathbf{R}$ and $0 < p \leq \infty$, then $\mathcal{L}^p(E)$ means $\mathcal{L}^p(\lambda_E)$, where λ_E denotes Lebesgue measure λ restricted to the Borel (or Lebesgue measurable) subsets of $\mathbf{R}$ that are contained in E.

With this convention, 7.10 implies that

$$\text{if } 0 < p < q < \infty, \text{ then } \mathcal{L}^q([0,1]) \subset \mathcal{L}^p([0,1]) \text{ and } \|f\|_p \leq \|f\|_q$$

for $f \in \mathcal{L}^q([0,1])$. See Exercises 12 and 13 in this section for related results.

Minkowski's Inequality

The next result is used as a tool to prove Minkowski's inequality (7.14). Once again, note the crucial role that Hölder's inequality plays in the proof.

7.12 *formula for* $\|f\|_p$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $1 \leq p < \infty$, and $f \in \mathcal{L}^p(\mu)$. Then

$$\|f\|_p = \sup\Bigl\{\Bigl|\int fh\,d\mu\Bigr| : h \in \mathcal{L}^{p'}(\mu) \text{ and } \|h\|_{p'} \leq 1\Bigr\}.$$

Proof If $\|f\|_p = 0$, then both sides of the equation in the conclusion of this result equal 0. Thus we assume that $\|f\|_p \neq 0$.

Hölder's inequality (7.9) implies that if $h \in \mathcal{L}^{p'}(\mu)$ and $\|h\|_{p'} \leq 1$, then

$$\Bigl|\int fh\,d\mu\Bigr| \leq \int |fh|\,d\mu \leq \|f\|_p\,\|h\|_{p'} \leq \|f\|_p.$$

Thus $\sup\bigl\{\bigl|\int fh\,d\mu\bigr| : h \in \mathcal{L}^{p'}(\mu) \text{ and } \|h\|_{p'} \leq 1\bigr\} \leq \|f\|_p$.

To prove the inequality in the other direction, define $h\colon X \to \mathbf{F}$ by

$$h(x) = \frac{\overline{f(x)}\,|f(x)|^{p-2}}{\|f\|_p^{p/p'}} \qquad (\text{set } h(x) = 0 \text{ when } f(x) = 0).$$

Then $\int fh\,d\mu = \|f\|_p$ and $\|h\|_{p'} = 1$, as you should verify (use $p - \frac{p}{p'} = 1$). Thus $\|f\|_p \leq \sup\bigl\{\bigl|\int fh\,d\mu\bigr| : h \in \mathcal{L}^{p'}(\mu) \text{ and } \|h\|_{p'} \leq 1\bigr\}$, as desired. ∎

7.13 Example *a point with infinite measure*

Suppose X is a set with exactly one element b and μ is the measure such that $\mu(\varnothing) = 0$ and $\mu(\{b\}) = \infty$. Then $\mathcal{L}^1(\mu)$ consists only of the 0 function. Thus if $p = \infty$ and f is the function whose value at b equals 1, then $\|f\|_\infty = 1$ but the right side of the equation in 7.12 equals 0. Thus 7.12 can fail when $p = \infty$.

Example 7.13 shows that we cannot take $p = \infty$ in 7.12. However, if μ is a σ-finite measure, then 7.12 holds even when $p = \infty$ (see Exercise 9).

The next result, which is called Minkowski's inequality, is an improvement for $p \geq 1$ of the inequality 7.5(a).

7.14 *Minkowski's inequality*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $1 \leq p \leq \infty$, and $f, g \in \mathcal{L}^p(\mu)$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof Assume that $1 \leq p < \infty$ (the case $p = \infty$ is left as an exercise for the reader). Inequality 7.5(a) implies that $f + g \in \mathcal{L}^p(\mu)$.

Suppose $h \in \mathcal{L}^{p'}(\mu)$ and $\|h\|_{p'} \leq 1$. Then

$$\begin{aligned}\Bigl|\int (f+g)h\,d\mu\Bigr| \leq \int |fh|\,d\mu + \int |gh|\,d\mu &\leq (\|f\|_p + \|g\|_p)\|h\|_{p'}\\ &\leq \|f\|_p + \|g\|_p,\end{aligned}$$

where the second inequality comes from Hölder's inequality (7.9). Now take the supremum of the left side of the inequality above over the set of $h \in \mathcal{L}^{p'}(\mu)$ such that $\|h\|_{p'} \leq 1$. By 7.12, we get $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, as desired. ∎

EXERCISES 7A

1 Suppose μ is a measure. Prove that

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \quad \text{and} \quad \|\alpha f\|_\infty = |\alpha|\,\|f\|_\infty$$

for all $f, g \in \mathcal{L}^\infty(\mu)$ and all $\alpha \in \mathbf{F}$. Conclude that with the usual operations of addition and scalar multiplication of functions, $\mathcal{L}^\infty(\mu)$ is a vector space.

2 Suppose $a \geq 0$, $b \geq 0$, and $1 < p < \infty$. Prove that

$$ab = \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

if and only if $a^p = b^{p'}$ [compare to Young's inequality (7.8)].

3 Suppose $a_1, \dots, a_n$ are nonnegative numbers. Prove that

$$(a_1 + \dots + a_n)^5 \leq n^4({a_1}^5 + \dots + {a_n}^5).$$

4 Prove Hölder's inequality (7.9) in the cases $p = 1$ and $p = \infty$.

5 Suppose that $(X, \mathcal{S}, \mu)$ is a measure space, $1 < p < \infty$, $f \in \mathcal{L}^p(\mu)$, and $h \in \mathcal{L}^{p'}(\mu)$. Prove that Hölder's inequality (7.9) is an equality if and only if there exist nonnegative numbers a and b, not both 0, such that

$$a|f(x)|^p = b|h(x)|^{p'}$$

for almost every $x \in X$.

6 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f \in \mathcal{L}^1(\mu)$, and $h \in \mathcal{L}^\infty(\mu)$. Prove that $\|fh\|_1 = \|f\|_1 \|h\|_\infty$ if and only if

$$|h(x)| = \|h\|_\infty$$

for almost every $x \in X$ such that $f(x) \neq 0$.

7 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, h \colon X \to \mathbf{F}$ are $\mathcal{S}$-measurable. Prove that

$$\|fh\|_r \leq \|f\|_p \|h\|_q$$

for all positive numbers p, q, r such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

8 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $n \in \mathbf{Z}^+$. Prove that

$$\|f_1 f_2 \cdots f_n\|_1 \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}$$

for all positive numbers $p_1, \ldots, p_n$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$ and all $\mathcal{S}$-measurable functions $f_1, f_2, \ldots, f_n \colon X \to \mathbf{F}$.

9 Show that the formula in 7.12 holds for $p = \infty$ if μ is a σ-finite measure.

10 Suppose $0 < p < q \leq \infty$.

(a) Prove that $\ell^p \subset \ell^q$.

(b) Prove that $\|(a_1, a_2, \ldots)\|_p \geq \|(a_1, a_2, \ldots)\|_q$ for every sequence $a_1, a_2, \ldots$ of elements of $\mathbf{F}$.

11 Show that $\bigcap\limits_{p>1} \ell^p \neq \ell^1$.

12 Show that $\bigcap\limits_{p<\infty} \mathcal{L}^p([0,1]) \neq \mathcal{L}^\infty([0,1])$.

13 Show that $\bigcup\limits_{p>1} \mathcal{L}^p([0,1]) \neq \mathcal{L}^1([0,1])$.

14 Suppose $p, q \in (0, \infty]$, with $p \neq q$. Prove that neither of the sets $\mathcal{L}^p(\mathbf{R})$ and $\mathcal{L}^q(\mathbf{R})$ is a subset of the other.

15 Show that there exists $f \in \mathcal{L}^2(\mathbf{R})$ such that $f \notin \mathcal{L}^p(\mathbf{R})$ for all $p \in (0, \infty] \setminus \{2\}$.

16 Suppose $(X, \mathcal{S}, \mu)$ is a finite measure space. Prove that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$$

for every $\mathcal{S}$-measurable function $f \colon X \to \mathbf{F}$.

17 Suppose μ is a measure, $0 < p \leq \infty$, and $f \in \mathcal{L}^p(\mu)$. Prove that for every $\varepsilon > 0$, there exists a simple function $g \in \mathcal{L}^p(\mu)$ such that $\|f - g\|_p < \varepsilon$. [*This exercise extends 3.44.*]

18 Suppose $0 < p < \infty$ and $f \in \mathcal{L}^p(\mathbf{R})$. Prove that for every $\varepsilon > 0$, there exists a step function $g \in \mathcal{L}^p(\mathbf{R})$ such that $\|f - g\|_p < \varepsilon$.
[*This exercise extends 3.47.*]

19 Suppose $0 < p < \infty$ and $f \in \mathcal{L}^p(\mathbf{R})$. Prove that for every $\varepsilon > 0$, there exists a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $\|f - g\|_p < \varepsilon$ and the set $\{x \in \mathbf{R} : g(x) \neq 0\}$ is bounded.
[*This exercise extends 3.48.*]

20 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $1 < p < \infty$, and $f, g \in \mathcal{L}^p(\mu)$. Prove that Minkowski's inequality (7.14) is an equality if and only if there exist nonnegative numbers a and b, not both 0, such that

$$af(x) = bg(x)$$

for almost every $x \in X$.

21 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, g \in \mathcal{L}^1(\mu)$. Prove that

$$\|f + g\|_1 = \|f\|_1 + \|g\|_1$$

if and only if $f(x)\overline{g(x)} \geq 0$ for almost every $x \in X$.

22 Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces and $0 < p < \infty$. Prove that if $f \in \mathcal{L}^p(\mu \times \nu)$, then

$$[f]_x \in \mathcal{L}^p(\nu) \text{ for almost every } x \in X$$

and

$$[f]^y \in \mathcal{L}^p(\mu) \text{ for almost every } y \in Y,$$

where $[f]_x$ and $[f]^y$ are the cross sections of f as defined in 5.7.

23 Suppose $1 \leq p < \infty$ and $f \in \mathcal{L}^p(\mathbf{R})$.

(a) For $t \in \mathbf{R}$, define $f_t\colon \mathbf{R} \to \mathbf{R}$ by $f_t(x) = f(x - t)$. Prove that the function $t \mapsto \|f - f_t\|_p$ is bounded and uniformly continuous on $\mathbf{R}$.

(b) For $t > 0$, define $f_t\colon \mathbf{R} \to \mathbf{R}$ by $f_t(x) = f(tx)$. Prove that

$$\lim_{t \to 1} \|f - f_t\|_p = 0.$$

24 Suppose $1 \leq p < \infty$ and $f \in \mathcal{L}^p(\mathbf{R})$. Prove that

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^p = 0$$

for almost every $b \in \mathbf{R}$.

7B $L^p(\mu)$

Definition of $L^p(\mu)$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $1 \le p \le \infty$. If there exists a nonempty set $E \in \mathcal{S}$ such that $\mu(E) = 0$, then $\|\chi_E\|_p = 0$ even though $\chi_E \neq 0$; thus $\|\cdot\|_p$ is not a norm on $\mathcal{L}^p(\mu)$. The standard way to deal with this problem is to identify functions that differ only on a set of μ-measure 0. To help make this process more rigorous, we introduce the following definitions.

7.15 Definition $\mathcal{Z}(\mu)$; $\tilde{f}$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 < p \le \infty$.

- $\mathcal{Z}(\mu)$ denotes the set of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ that equal 0 almost everywhere.
- For $f \in \mathcal{L}^p(\mu)$, let $\tilde{f}$ be the subset of $\mathcal{L}^p(\mu)$ defined by
$$\tilde{f} = \{f + z : z \in \mathcal{Z}(\mu)\}.$$

The set $\mathcal{Z}(\mu)$ is clearly closed under scalar multiplication. Also, $\mathcal{Z}(\mu)$ is closed under addition because the union of two sets with μ-measure 0 is a set with μ-measure 0. Thus $\mathcal{Z}(\mu)$ is a subspace of $\mathcal{L}^p(\mu)$, as we had noted in the third bullet point of Example 6.32.

Note that if $f, F \in \mathcal{L}^p(\mu)$, then $\tilde{f} = \tilde{F}$ if and only if $f(x) = F(x)$ for almost every $x \in X$.

7.16 Definition $L^p(\mu)$

Suppose μ is a measure and $0 < p \le \infty$.

- Let $L^p(\mu)$ denote the collection of subsets of $\mathcal{L}^p(\mu)$ defined by
$$L^p(\mu) = \{\tilde{f} : f \in \mathcal{L}^p(\mu)\}.$$
- For $\tilde{f}, \tilde{g} \in L^p(\mu)$ and $\alpha \in \mathbf{F}$, define $\tilde{f} + \tilde{g}$ and $\alpha \tilde{f}$ by
$$\tilde{f} + \tilde{g} = (f + g)\tilde{} \quad \text{and} \quad \alpha \tilde{f} = (\alpha f)\tilde{}.$$

The last bullet point in the definition above requires a bit of care to verify that it makes sense. The potential problem is that if $\mathcal{Z}(\mu) \neq \{0\}$, then $\tilde{f}$ is not uniquely represented by f. Thus suppose $f, F, g, G \in \mathcal{L}^p(\mu)$ and $\tilde{f} = \tilde{F}$ and $\tilde{g} = \tilde{G}$. For the definition of addition in $L^p(\mu)$ to make sense, we must verify that $(f + g)\tilde{} = (F + G)\tilde{}$. This verification is left to the reader, as is the similar verification that the scalar multiplication defined in the last bullet point above makes sense.

You might want to think of elements of $L^p(\mu)$ as equivalence classes of functions in $\mathcal{L}^p(\mu)$, where two functions are equivalent if they agree almost everywhere.

Mathematicians often pretend that elements of $L^p(\mu)$ are functions, where two functions are considered to be equal if they differ only on a set of μ-measure 0. This fiction is harmless provided that the operations you perform with such "functions" produce the same results if the functions are changed on a set of measure 0.

Note the subtle typographic difference between $\mathcal{L}^p(\mu)$ and $L^p(\mu)$. An element of the calligraphic $\mathcal{L}^p(\mu)$ is a function; an element of the italic $L^p(\mu)$ is a set of functions, any two of which agree almost everywhere.

7.17 Definition $\|\cdot\|_p$ ***on*** $L^p(\mu)$

Suppose μ is a measure and $0 < p \leq \infty$. Define $\|\cdot\|_p$ on $L^p(\mu)$ by

$$\|\tilde{f}\|_p = \|f\|_p$$

for $f \in \mathcal{L}^p(\mu)$.

Note that if $f, F \in \mathcal{L}^p(\mu)$ and $\tilde{f} = \tilde{F}$, then $\|f\|_p = \|F\|_p$. Thus the definition above makes sense.

In the result below, the addition and scalar multiplication on $L^p(\mu)$ come from 7.16 and the norm comes from 7.17.

7.18 $L^p(\mu)$ ***is a normed vector space***

Suppose μ is a measure and $1 \leq p \leq \infty$. Then $L^p(\mu)$ is a vector space and $\|\cdot\|_p$ is a norm on $L^p(\mu)$.

The proof of the result above is left to the reader, who will surely use Minkowski's inequality (7.14) to verify the triangle inequality. Note that the additive identity of $L^p(\mu)$ is $\tilde{0}$, which equals $\mathcal{Z}(\mu)$.

For readers familiar with quotients of vector spaces: you may recognize that $L^p(\mu)$ is the quotient space

$$\mathcal{L}^p(\mu)/\mathcal{Z}(\mu).$$

For readers who want to learn about quotients of vector spaces: see a textbook for a second course in linear algebra.

If μ is counting measure on $\mathbf{Z}^+$, then

$$\mathcal{L}^p(\mu) = L^p(\mu) = \ell^p$$

because counting measure has no sets of measure 0 other than the empty set.

In the next definition, note that if E is a Borel set then 2.95 implies $L^p(E)$ using Borel measurable functions equals $L^p(E)$ using Lebesgue measurable functions.

7.19 Definition $L^p(E)$ ***for*** $E \subset \mathbf{R}$

If E is a Borel (or Lebesgue measurable) subset of $\mathbf{R}$ and $0 < p \leq \infty$, then $L^p(E)$ means $L^p(\lambda_E)$, where λ_E denotes Lebesgue measure λ restricted to the Borel (or Lebesgue measurable) subsets of $\mathbf{R}$ that are contained in E.

$L^p(\mu)$ Is a Banach Space

The proof of the next result does all the hard work we need to prove that $L^p(\mu)$ is a Banach space. However, we state the next result in terms of $\mathcal{L}^p(\mu)$ instead of $L^p(\mu)$ so that we can work with genuine functions. Moving to $L^p(\mu)$ will then be easy (see 7.24).

7.20 *Cauchy sequences in $\mathcal{L}^p(\mu)$ converge*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $1 \le p \le \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of functions in $\mathcal{L}^p(\mu)$ such that for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that

$$\|f_j - f_k\|_p < \varepsilon$$

for all $j \ge n$ and $k \ge n$. Then there exists $f \in \mathcal{L}^p(\mu)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_p = 0.$$

Proof The case $p = \infty$ is left as an exercise for the reader. Thus assume $1 \le p < \infty$.

It suffices to show that $\lim_{m\to\infty} \|f_{k_m} - f\|_p = 0$ for some $f \in \mathcal{L}^p(\mu)$ and some subsequence $f_{k_1}, f_{k_2}, \ldots$ (see Exercise 14 of Section 6A, whose proof does not require the positive definite property of a norm).

Thus dropping to a subsequence (but not relabeling) and setting $f_0 = 0$, we can assume that

$$\sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_p < \infty.$$

Define functions $g_1, g_2, \ldots$ and g from X to $[0, \infty]$ by

$$g_m(x) = \sum_{k=1}^{m} |f_k(x) - f_{k-1}(x)| \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} |f_k(x) - f_{k-1}(x)|.$$

Minkowski's inequality (7.14) implies that

7.21
$$\|g_m\|_p \le \sum_{k=1}^{m} \|f_k - f_{k-1}\|_p.$$

Clearly $\lim_{m\to\infty} g_m(x) = g(x)$ for every $x \in X$. Thus the Monotone Convergence Theorem (3.11) and 7.21 imply

7.22
$$\int g^p \, d\mu = \lim_{m\to\infty} \int {g_m}^p \, d\mu \le \Bigl(\sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_p\Bigr)^p < \infty.$$

Thus $g(x) < \infty$ for almost every $x \in X$.

Because every infinite series of real numbers that converges absolutely also converges, for almost every $x \in X$ we can define $f(x)$ by

$$f(x) = \sum_{k=1}^{\infty} \bigl(f_k(x) - f_{k-1}(x)\bigr) = \lim_{m\to\infty} \sum_{k=1}^{m} \bigl(f_k(x) - f_{k-1}(x)\bigr) = \lim_{m\to\infty} f_m(x).$$

In particular, $\lim_{m\to\infty} f_m(x)$ exists for almost every $x \in X$. Define $f(x)$ to be 0 for those $x \in X$ for which the limit does not exist.

We now have a function f that is the pointwise limit (almost everywhere) of $f_1, f_2, \ldots$. The definition of f shows that $|f(x)| \le g(x)$ for almost every $x \in X$. Thus 7.22 shows that $f \in \mathcal{L}^p(\mu)$.

To show that $\lim_{k\to\infty} \|f_k - f\|_p = 0$, suppose $\varepsilon > 0$ and let $n \in \mathbf{Z}^+$ be such that $\|f_j - f_k\|_p < \varepsilon$ for all $j \ge n$ and $k \ge n$. Suppose $k \ge n$. Then

$$\begin{aligned}
\|f_k - f\|_p &= \Big(\int |f_k - f|^p \, d\mu\Big)^{1/p} \\
&\le \liminf_{j\to\infty} \Big(\int |f_k - f_j|^p \, d\mu\Big)^{1/p} \\
&= \liminf_{j\to\infty} \|f_k - f_j\|_p \\
&\le \varepsilon,
\end{aligned}$$

where the second line above comes from Fatou's Lemma (Exercise 17 in Section 3A). Thus $\lim_{k\to\infty} \|f_k - f\|_p = 0$, as desired. ■

The proof that we have just completed contains within it the proof of a useful result that is worth stating separately. A sequence can converge in p-norm without converging pointwise anywhere (see, for example, Exercise 12). However, the next result guarantees that some subsequence converges pointwise almost everywhere.

7.23 *convergent sequences in $\mathcal{L}^p$ have pointwise convergent subsequences*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $1 \le p \le \infty$. Suppose $f \in \mathcal{L}^p(\mu)$ and $f_1, f_2, \ldots$ is a sequence of functions in $\mathcal{L}^p(\mu)$ such that $\lim_{k\to\infty} \|f_k - f\|_p = 0$. Then there exists a subsequence $f_{k_1}, f_{k_2}, \ldots$ such that

$$\lim_{m\to\infty} f_{k_m}(x) = f(x)$$

for almost every $x \in X$.

Proof Suppose $f_{k_1}, f_{k_2}, \ldots$ is a subsequence such that

$$\sum_{m=2}^{\infty} \|f_{k_m} - f_{k_{m-1}}\|_p < \infty.$$

An examination of the proof of 7.20 shows that $\lim_{m\to\infty} f_{k_m}(x) = f(x)$ for almost every $x \in X$. ■

7.24 *$L^p(\mu)$ is a Banach space*

Suppose μ is a measure and $1 \le p \le \infty$. Then $L^p(\mu)$ is a Banach space.

Proof This result follows immediately from 7.20 and the appropriate definitions. ■

Duality

Recall that the dual space of a normed vector space V is denoted by V' and is defined to be the Banach space of bounded linear functionals on V (see 6.71).

In the statement and proof of the next result, an element of an L^p space is denoted by a symbol that makes it look like a function rather than like a collection of functions that agree except on a set of measure 0. However, because integrals and L^p-norms are unchanged when functions change only on a set of measure 0, this notational convenience causes no problems.

7.25 *natural map of $L^{p'}(\mu)$ into $\big(L^p(\mu)\big)'$ preserves norms*

Suppose μ is a measure and $1 < p \le \infty$. For $h \in L^{p'}(\mu)$, define $\varphi_h \colon L^p(\mu) \to \mathbf{F}$ by

$$\varphi_h(f) = \int fh\,d\mu.$$

Then $h \mapsto \varphi_h$ is a one-to-one linear map from $L^{p'}(\mu)$ to $\big(L^p(\mu)\big)'$. Furthermore, $\|\varphi_h\| = \|h\|_{p'}$ for all $h \in L^{p'}(\mu)$.

Proof Suppose $h \in L^{p'}(\mu)$ and $f \in L^p(\mu)$. Then Hölder's inequality (7.9) tells us that $fh \in L^1(\mu)$ and that

$$\|fh\|_1 \le \|h\|_{p'}\|f\|_p.$$

Thus φ_h, as defined above, is a bounded linear map from $L^p(\mu)$ to $\mathbf{F}$. Also, the map $h \mapsto \varphi_h$ is clearly a linear map of $L^{p'}(\mu)$ into $\big(L^p(\mu)\big)'$. Now 7.12 (with the roles of p and p' reversed) shows that

$$\|\varphi_h\| = \sup\{|\varphi_h(f)| : f \in L^p(\mu) \text{ and } \|f\|_p \le 1\} = \|h\|_{p'}.$$

If $h_1, h_2 \in L^{p'}(\mu)$ and $\varphi_{h_1} = \varphi_{h_2}$, then

$$\|h_1 - h_2\|_{p'} = \|\varphi_{h_1-h_2}\| = \|\varphi_{h_1} - \varphi_{h_2}\| = \|0\| = 0,$$

which implies $h_1 = h_2$. Thus $h \mapsto \varphi_h$ is a one-to-one map from $L^{p'}(\mu)$ to $\big(L^p(\mu)\big)'$. ■

The result in 7.25 fails for some measures μ if $p = 1$. However, if μ is a σ-finite measure, then 7.25 holds even if $p = 1$ (see Exercise 14).

Is the range of the map $h \mapsto \varphi_h$ in 7.25 all of $\big(L^p(\mu)\big)'$? The next result provides an affirmative answer to this question in the special case of ℓ^p for $1 \le p < \infty$. We will deal with this question for more general measures later (see 9.42; also see Exercise 25 in Section 8B).

When thinking of ℓ^p as a normed vector space, as in the next result, unless stated otherwise you should always assume that the norm on ℓ^p is the usual norm $\|\cdot\|_p$ that is associated with $\mathcal{L}^p(\mu)$, where μ is counting measure on $\mathbf{Z}^+$. In other words, if $1 \le p < \infty$, then

$$\|(a_1, a_2, \ldots)\|_p = \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{1/p}.$$

7.26 *dual space of ℓ^p can be identified with $\ell^{p'}$*

Suppose $1 \le p < \infty$. For $b = (b_1, b_2, \ldots) \in \ell^{p'}$, define $\varphi_b \colon \ell^p \to \mathbf{F}$ by

$$\varphi_b(a) = \sum_{k=1}^{\infty} a_k b_k,$$

where $a = (a_1, a_2, \ldots)$. Then $b \mapsto \varphi_b$ is a one-to-one linear map from $\ell^{p'}$ onto $(\ell^p)'$. Furthermore, $\|\varphi_b\| = \|b\|_{p'}$ for all $b \in \ell^{p'}$.

Proof For $k \in \mathbf{Z}^+$, let $e_k \in \ell^p$ be the sequence in which each term is 0 except that the kth term is 1; thus $e_k = (0, \ldots, 0, 1, 0, \ldots)$.

Suppose $\varphi \in (\ell^p)'$. Define a sequence $b = (b_1, b_2, \ldots)$ of numbers in $\mathbf{F}$ by

$$b_k = \varphi(e_k).$$

Suppose $a = (a_1, a_2, \ldots) \in \ell^p$. Then

$$a = \sum_{k=1}^{\infty} a_k e_k,$$

where the infinite sum converges in the norm of ℓ^p (the proof would fail here if we allowed p to be ∞). Because φ is a bounded linear functional on ℓ^p, applying φ to both sides of the equation above shows that

$$\varphi(a) = \sum_{k=1}^{\infty} a_k b_k.$$

We still need to prove that $b \in \ell^{p'}$. To do this, for $n \in \mathbf{Z}^+$ let μ_n be counting measure on $\{1, 2, \ldots, n\}$. We can think of $L^p(\mu_n)$ as a subspace of ℓ^p by identifying each $(a_1, \ldots, a_n) \in L^p(\mu_n)$ with $(a_1, \ldots, a_n, 0, 0, \ldots) \in \ell^p$. Restricting the linear functional φ to $L^p(\mu_n)$ gives the linear functional on $L^p(\mu_n)$ that satisfies the following equation:

$$\varphi|_{L^p(\mu_n)}(a_1, \ldots, a_n) = \sum_{k=1}^{n} a_k b_k.$$

Now 7.25 (also see Exercise 14 for the case where $p = 1$) gives

$$\begin{aligned} \|(b_1, \ldots, b_n)\|_{p'} &= \|\varphi|_{L^p(\mu_n)}\| \\ &\le \|\varphi\|. \end{aligned}$$

Because $\lim_{n\to\infty} \|(b_1, \ldots, b_n)\|_{p'} = \|b\|_{p'}$, the inequality above implies the inequality $\|b\|_{p'} \le \|\varphi\|$. Thus $b \in \ell^{p'}$, which implies that $\varphi = \varphi_b$, completing the proof. ■

The previous result does not hold when $p = \infty$. In other words, the dual space of ℓ^∞ cannot be identified with ℓ^1. However, see Exercise 15, which shows that the dual space of a natural subspace of ℓ^∞ can be identified with ℓ^1.

EXERCISES 7B

1 Suppose $n > 1$ and $0 < p < 1$. Prove that if $\|\cdot\|$ is defined on $\mathbf{F}^n$ by

$$\|(a_1, \ldots, a_n)\| = \left(|a_1|^p + \cdots + |a_n|^p\right)^{1/p},$$

then $\|\cdot\|$ is not a norm on $\mathbf{F}^n$.

2 (a) Suppose $1 \leq p < \infty$. Prove that there is a countable subset of ℓ^p whose closure equals ℓ^p.

(b) Prove that there does not exist a countable subset of ℓ^∞ whose closure equals ℓ^∞.

3 (a) Suppose $1 \leq p < \infty$. Prove that there is a countable subset of $L^p(\mathbf{R})$ whose closure equals $L^p(\mathbf{R})$.

(b) Prove that there does not exist a countable subset of $L^\infty(\mathbf{R})$ whose closure equals $L^\infty(\mathbf{R})$.

4 Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $1 \leq p \leq \infty$. Prove that if $f\colon X \to \mathbf{F}$ is an $\mathcal{S}$-measurable function such that $fh \in \mathcal{L}^1(\mu)$ for every $h \in \mathcal{L}^{p'}(\mu)$, then $f \in \mathcal{L}^p(\mu)$.

5 (a) Prove that if μ is a measure, $1 < p < \infty$, and $f, g \in L^p(\mu)$ are such that

$$\|f\|_p = \|g\|_p = \left\|\frac{f+g}{2}\right\|_p,$$

then $f = g$.

(b) Give an example to show that the result in part (a) can fail if $p = 1$.

(c) Give an example to show that the result in part (a) can fail if $p = \infty$.

6 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 < p < 1$. Show that

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

for all $\mathcal{S}$-measurable functions $f, g\colon X \to \mathbf{F}$.

7 Prove that $L^p(\mu)$, with addition and scalar multiplication as defined in 7.16 and norm defined as in 7.17, is a normed vector space. In other words, prove 7.18.

8 Prove 7.20 for the case $p = \infty$.

9 Prove that 7.20 also holds for $p \in (0, 1)$.

10 Prove that 7.23 also holds for $p \in (0, 1)$.

11 Suppose $1 \leq p \leq \infty$. Prove that

$$\{(a_1, a_2, \ldots) \in \ell^p : a_k \neq 0 \text{ for every } k \in \mathbf{Z}^+\}$$

is not an open subset of ℓ^p.

12 Show that there exists a sequence $f_1, f_2, \dots$ of functions in $\mathcal{L}^1([0,1])$ such that $\lim_{k\to\infty} \|f_k\|_1 = 0$ but

$$\sup\{f_k(x) : k \in \mathbf{Z}^+\} = \infty$$

for every $x \in [0,1]$.

[*This exercise shows that the conclusion of 7.23 cannot be improved to conclude that* $\lim_{k\to\infty} f_k(x) = f(x)$ *for almost every* $x \in X$.]

13 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $1 \le p \le \infty$, $f \in \mathcal{L}^p(\mu)$, and $f_1, f_2, \dots$ is a sequence in $\mathcal{L}^p(\mu)$ such that $\lim_{k\to\infty} \|f_k - f\|_p = 0$. Show that if $g\colon X \to \mathbf{F}$ is a function such that $\lim_{k\to\infty} f_k(x) = g(x)$ for almost every $x \in X$, then $f(x) = g(x)$ for almost every $x \in X$.

14 (a) Give an example of a measure μ such that 7.25 fails for $p = 1$.

(b) Show that if μ is a σ-finite measure, then 7.25 holds for $p = 1$.

15 Let

$$c_0 = \{(a_1, a_2, \dots) \in \ell^\infty : \lim_{k\to\infty} a_k = 0\}.$$

Give c_0 the norm that it inherits as a subspace of ℓ^∞.

(a) Prove that c_0 is a Banach space.

(b) Prove that the dual space of c_0 can be identified with ℓ^1.

16 Suppose $1 \le p \le 2$.

(a) Prove that if $w, z \in \mathbf{C}$, then

$$\frac{|w+z|^p + |w-z|^p}{2} \le |w|^p + |z|^p \le \frac{|w+z|^p + |w-z|^p}{2^{p-1}}.$$

(b) Prove that if μ is a measure and $f, g \in \mathcal{L}^p(\mu)$, then

$$\frac{\|f+g\|_p^p + \|f-g\|_p^p}{2} \le \|f\|_p^p + \|g\|_p^p \le \frac{\|f+g\|_p^p + \|f-g\|_p^p}{2^{p-1}}.$$

17 Suppose $2 \le p < \infty$.

(a) Prove that if $w, z \in \mathbf{C}$, then

$$\frac{|w+z|^p + |w-z|^p}{2^{p-1}} \le |w|^p + |z|^p \le \frac{|w+z|^p + |w-z|^p}{2}.$$

(b) Prove that if μ is a measure and $f, g \in \mathcal{L}^p(\mu)$, then

$$\frac{\|f+g\|_p^p + \|f-g\|_p^p}{2^{p-1}} \le \|f\|_p^p + \|g\|_p^p \le \frac{\|f+g\|_p^p + \|f-g\|_p^p}{2}.$$

[*The inequalities in the two previous exercises are called* Clarkson's inequalities. *They were discovered by James Clarkson in 1936.*]

18 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $1 \le p, q \le \infty$, and $h \colon X \to \mathbf{F}$ is an $\mathcal{S}$-measurable function such that $hf \in L^q(\mu)$ for every $f \in L^p(\mu)$. Prove that $f \mapsto hf$ is a continuous linear map from $L^p(\mu)$ to $L^q(\mu)$.

***A Banach space is called* reflexive *if the canonical isometry of the Banach space into its double dual space is surjective* (*see Exercise 20 in Section 6D for the definitions of the double dual space and the canonical isometry*).**

19 Prove that if $1 < p < \infty$, then ℓ^p is reflexive.

20 Prove that ℓ^1 is not reflexive.

21 Show that with the natural identifications, the canonical isometry of c_0 into its double dual space is the inclusion map of c_0 into ℓ^∞ (see Exercise 15 for the definition of c_0 and an identification of its dual space).

22 Suppose $1 \le p < \infty$ and V, W are Banach spaces. Show that $V \times W$ is a Banach space if the norm on $V \times W$ is defined by

$$\|(f, g)\| = \left(\|f\|^p + \|g\|^p\right)^{1/p}$$

for $f \in V$ and $g \in W$.

Chapter 8

Hilbert Spaces

Normed vector spaces and Banach spaces, which were introduced in Chapter 6, capture the notion of distance. In this chapter we introduce inner product spaces, which capture the notion of angle. The concept of orthogonality, which corresponds to right angles in the familiar context of $\mathbf{R}^2$ or $\mathbf{R}^3$, plays a particularly important role in inner product spaces.

Just as a Banach space is defined to be a normed vector space in which every Cauchy sequence converges, a Hilbert space is defined to be an inner product space that is a Banach space. Hilbert spaces are named in honor of David Hilbert (1862–1943), who helped develop parts of the theory in the early twentieth century.

In this chapter, we will see a clean description of the bounded linear functionals on a Hilbert space. We will also see that every Hilbert space has an orthonormal basis, which make Hilbert spaces look much like standard Euclidean spaces but with infinite sums replacing finite sums.



The Mathematical Institute at the University of Göttingen, Germany. This building was opened in 1930, when Hilbert was near the end of his career there. Other prominent mathematicians who taught at the University of Göttingen and made major contributions to mathematics include Richard Courant (1888–1972), Richard Dedekind (1831–1916), Gustav Lejeune Dirichlet (1805–1859), Carl Friedrich Gauss (1777–1855), Hermann Minkowski (1864–1909), Emmy Noether (1882–1935), and Bernhard Riemann (1826–1866).

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_8

8A Inner Product Spaces

Inner Products

If $p = 2$, then the dual exponent p' also equals 2. In this special case Hölder's inequality (7.9) implies that if μ is a measure, then

$$\left|\int fg\,d\mu\right| \leq \|f\|_2\|g\|_2$$

for all $f, g \in \mathcal{L}^2(\mu)$. Thus we can associate with each pair of functions $f, g \in \mathcal{L}^2(\mu)$ a number $\int fg\,d\mu$. An inner product is almost a generalization of this pairing, with a slight twist to get a closer connection to the $L^2(\mu)$-norm.

If $g = f$ and $\mathbf{F} = \mathbf{R}$, then the left side of the inequality above is $\|f\|_2^2$. However, if $g = f$ and $\mathbf{F} = \mathbf{C}$, then the left side of the inequality above need not equal $\|f\|_2^2$. Instead, we should take $g = \overline{f}$ to get $\|f\|_2^2$ above.

The observations above suggest that we should consider the pairing that takes f, g to $\int f\,\overline{g}\,d\mu$. Then pairing f with itself gives $\|f\|_2^2$.

Now we are ready to define inner products, which abstract the key properties of the pairing $f, g \mapsto \int f\,\overline{g}\,d\mu$ on $L^2(\mu)$, where μ is a measure.

8.1 Definition ***inner product; inner product space***

An *inner product* on a vector space V is a function that takes each ordered pair f, g of elements of V to a number $\langle f, g\rangle \in \mathbf{F}$ and has the following properties:

- **positivity**
  $\langle f, f\rangle \in [0, \infty)$ for all $f \in V$;
- **definiteness**
  $\langle f, f\rangle = 0$ if and only if $f = 0$;
- **linearity in first slot**
  $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ and $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$ for all $f, g, h \in V$ and all $\alpha \in \mathbf{F}$;
- **conjugate symmetry**
  $\langle f, g\rangle = \overline{\langle g, f\rangle}$ for all $f, g \in V$.

A vector space with an inner product on it is called an *inner product space*. The terminology *real inner product space* indicates that $\mathbf{F} = \mathbf{R}$; the terminology *complex inner product space* indicates that $\mathbf{F} = \mathbf{C}$.

If $\mathbf{F} = \mathbf{R}$, then the complex conjugate above can be ignored and the conjugate symmetry property above can be rewritten more simply as $\langle f, g\rangle = \langle g, f\rangle$ for all $f, g \in V$.

Although most mathematicians define an inner product as above, many physicists use a definition that requires linearity in the second slot instead of the first slot.

8.2 Example *inner product spaces*

- For $n \in \mathbf{Z}^+$, define an inner product on $\mathbf{F}^n$ by
$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1 \overline{b_1} + \cdots + a_n \overline{b_n}$$
for $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbf{F}^n$. When thinking of $\mathbf{F}^n$ as an inner product space, we always mean this inner product unless the context indicates some other inner product.

- Define an inner product on ℓ^2 by
$$\langle (a_1, a_2, \ldots), (b_1, b_2, \ldots) \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$$
for $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in \ell^2$. Hölder's inequality (7.9), as applied to counting measure on $\mathbf{Z}^+$ and taking $p = 2$, implies that the infinite sum above converges absolutely and hence converges to an element of $\mathbf{F}$. When thinking of ℓ^2 as an inner product space, we always mean this inner product unless the context indicates some other inner product.

- Define an inner product on $C([0,1])$, which is the vector space of continuous functions from $[0,1]$ to $\mathbf{F}$, by
$$\langle f, g \rangle = \int_0^1 f \, \overline{g}$$
for $f, g \in C([0,1])$. The definiteness requirement for an inner product is satisfied because if $f \colon [0,1] \to \mathbf{F}$ is a continuous function such that $\int_0^1 |f|^2 = 0$, then the function f is identically 0.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Define an inner product on $L^2(\mu)$ by
$$\langle f, g \rangle = \int f \, \overline{g} \, d\mu$$
for $f, g \in L^2(\mu)$. Hölder's inequality (7.9) with $p = 2$ implies that the integral above makes sense as an element of $\mathbf{F}$. When thinking of $L^2(\mu)$ as an inner product space, we always mean this inner product unless the context indicates some other inner product.

  Here we use $L^2(\mu)$ rather than $\mathcal{L}^2(\mu)$ because the definiteness requirement fails on $\mathcal{L}^2(\mu)$ if there exist nonempty sets $E \in \mathcal{S}$ such that $\mu(E) = 0$ (consider $\langle \chi_E, \chi_E \rangle$ to see the problem).

  The first two bullet points in this example are special cases of $L^2(\mu)$, taking μ to be counting measure on either $\{1, \ldots, n\}$ or $\mathbf{Z}^+$.

As we will see, even though the main examples of inner product spaces are $L^2(\mu)$ spaces, working with the inner product structure is often cleaner and simpler than working with measures and integrals.

8.3 *basic properties of an inner product*

Suppose V is an inner product space. Then

(a) $\langle 0, g\rangle = \langle g, 0\rangle = 0$ for every $g \in V$;

(b) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$ for all $f, g, h \in V$;

(c) $\langle f, \alpha g\rangle = \overline{\alpha}\langle f, g\rangle$ for all $\alpha \in \mathbf{F}$ and $f, g \in V$.

Proof

(a) For $g \in V$, the function $f \mapsto \langle f, g\rangle$ is a linear map from V to $\mathbf{F}$. Because every linear map takes 0 to 0, we have $\langle 0, g\rangle = 0$. Now the conjugate symmetry property of an inner product implies that

$$\langle g, 0\rangle = \overline{\langle 0, g\rangle} = \overline{0} = 0.$$

(b) Suppose $f, g, h \in V$. Then

$$\langle f, g+h\rangle = \overline{\langle g+h, f\rangle} = \overline{\langle g, f\rangle + \langle h, f\rangle} = \overline{\langle g, f\rangle} + \overline{\langle h, f\rangle} = \langle f, g\rangle + \langle f, h\rangle.$$

(c) Suppose $\alpha \in \mathbf{F}$ and $f, g \in V$. Then

$$\langle f, \alpha g\rangle = \overline{\langle \alpha g, f\rangle} = \overline{\alpha\langle g, f\rangle} = \overline{\alpha}\,\overline{\langle g, f\rangle} = \overline{\alpha}\langle f, g\rangle,$$

as desired. ■

If $\mathbf{F} = \mathbf{R}$, then parts (b) and (c) of 8.3 imply that for $f \in V$, the function $g \mapsto \langle f, g\rangle$ is a linear map from V to $\mathbf{R}$. However, if $\mathbf{F} = \mathbf{C}$ and $f \neq 0$, then the function $g \mapsto \langle f, g\rangle$ is not a linear map from V to $\mathbf{C}$ because of the complex conjugate in part (c) of 8.3.

Cauchy–Schwarz Inequality and Triangle Inequality

Now we can define the norm associated with each inner product. We use the word *norm* (which will turn out to be correct) even though it is not yet clear that all the properties required of a norm are satisfied.

8.4 Definition *norm associated with an inner product;* $\|\cdot\|$

Suppose V is an inner product space. For $f \in V$, define the *norm* of f, denoted $\|f\|$, by

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

8.5 **Example** *norms on inner product spaces*

In each of the following examples, the inner product is the standard inner product as defined in Example 8.2.

- If $n \in \mathbf{Z}^+$ and $(a_1, \ldots, a_n) \in \mathbf{F}^n$, then

$$\|(a_1, \ldots, a_n)\| = \sqrt{|a_1|^2 + \cdots + |a_n|^2}.$$

  Thus the norm on $\mathbf{F}^n$ associated with the standard inner product is the usual Euclidean norm.

- If $(a_1, a_2, \ldots) \in \ell^2$, then

$$\|(a_1, a_2, \ldots)\| = \Bigl(\sum_{k=1}^{\infty} |a_k|^2\Bigr)^{1/2}.$$

  Thus the norm associated with the inner product on ℓ^2 is just the standard norm $\|\cdot\|_2$ on ℓ^2 as defined in Example 7.2.

- If μ is a measure and $f \in L^2(\mu)$, then

$$\|f\| = \Bigl(\int |f|^2 \, d\mu\Bigr)^{1/2}.$$

  Thus the norm associated with the inner product on $L^2(\mu)$ is just the standard norm $\|\cdot\|_2$ on $L^2(\mu)$ as defined in 7.17.

The definition of an inner product (8.1) implies that if V is an inner product space and $f \in V$, then

- $\|f\| \geq 0$;
- $\|f\| = 0$ if and only if $f = 0$.

The proof of the next result illustrates a frequently used property of the norm on an inner product space: working with the square of the norm is often easier than working directly with the norm.

8.6 ***homogeneity of the norm***

Suppose V is an inner product space, $f \in V$, and $\alpha \in \mathbf{F}$. Then

$$\|\alpha f\| = |\alpha| \, \|f\|.$$

Proof We have

$$\|\alpha f\|^2 = \langle \alpha f, \alpha f \rangle = \alpha \langle f, \alpha f \rangle = \alpha \overline{\alpha} \langle f, f \rangle = |\alpha|^2 \|f\|^2.$$

Taking square roots now gives the desired equality. ∎

The next definition plays a crucial role in the study of inner product spaces.

8.7 Definition ***orthogonal***

Two elements of an inner product space are called *orthogonal* if their inner product equals 0.

In the definition above, the order of the two elements of the inner product space does not matter because $\langle f, g\rangle = 0$ if and only if $\langle g, f\rangle = 0$. Instead of saying that f and g are orthogonal, sometimes we say that f is orthogonal to g.

8.8 Example *orthogonal elements of an inner product space*

- In $\mathbf{C}^3$, $(2, 3, 5i)$ and $(6, 1, -3i)$ are orthogonal because

$$\langle (2,3,5i),(6,1,-3i)\rangle = 2\cdot 6 + 3\cdot 1 + 5i\cdot(3i) = 12 + 3 - 15 = 0.$$

- The elements of $L^2\big((-\pi,\pi]\big)$ represented by $\sin(3t)$ and $\cos(8t)$ are orthogonal because

$$\int_{-\pi}^{\pi} \sin(3t)\cos(8t)\,dt = \Big[\frac{\cos(5t)}{10} - \frac{\cos(11t)}{22}\Big]_{t=-\pi}^{t=\pi} = 0,$$

where dt denotes integration with respect to Lebesgue measure on $(-\pi,\pi]$.

Exercise 8 asks you to prove that if a and b are nonzero elements in $\mathbf{R}^2$, then

$$\langle a, b\rangle = \|a\|\,\|b\|\cos\theta,$$

where θ is the angle between a and b (thinking of a as the vector whose initial point is the origin and whose end point is a, and similarly for b). Thus two elements of $\mathbf{R}^2$ are orthogonal if and only if the cosine of the angle between them is 0, which happens if and only if the vectors are perpendicular in the usual sense of plane geometry. Thus you can think of the word *orthogonal* as a fancy word meaning *perpendicular*.

Law professor Richard Friedman presenting a case before the U.S. Supreme Court in 2010:

Mr. Friedman: I think that issue is entirely orthogonal to the issue here because the Commonwealth is acknowledging—
Chief Justice Roberts: I'm sorry. Entirely what?
Mr. Friedman: Orthogonal. Right angle. Unrelated. Irrelevant.
Chief Justice Roberts: Oh.
Justice Scalia: What was that adjective? I liked that.
Mr. Friedman: Orthogonal.
Chief Justice Roberts: Orthogonal.
Mr. Friedman: Right, right.
Justice Scalia: Orthogonal, ooh. (Laughter.)
Justice Kennedy: I knew this case presented us a problem. (Laughter.)

The next theorem is over 2500 years old, although it was not originally stated in the context of inner product spaces.

8.9 *Pythagorean Theorem*

Suppose f and g are orthogonal elements of an inner product space. Then

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2.$$

Proof We have

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle \\ &= \|f\|^2 + \|g\|^2,\end{aligned}$$

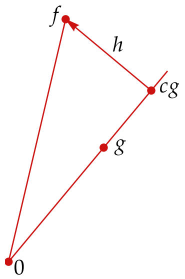
as desired. ■

Exercise 3 shows that whether or not the converse of the Pythagorean Theorem holds depends upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

Suppose f and g are elements of an inner product space V, with $g \neq 0$. Frequently it is useful to write f as some number c times g plus an element h of V that is orthogonal to g. The figure here suggests that such a decomposition should be possible. To find the appropriate choice for c, note that if $f = cg + h$ for some $c \in \mathbf{F}$ and some $h \in V$ with $\langle h, g\rangle = 0$, then we must have

$$\langle f, g\rangle = \langle cg + h, g\rangle = c\|g\|^2,$$

which implies that $c = \dfrac{\langle f,g\rangle}{\|g\|^2}$, which then implies that $h = f - \dfrac{\langle f,g\rangle}{\|g\|^2}g$. Hence we are led to the following result.



Here $f = cg + h$, where h is orthogonal to g.

8.10 *orthogonal decomposition*

Suppose f and g are elements of an inner product space, with $g \neq 0$. Then there exists $h \in V$ such that

$$\langle h, g\rangle = 0 \quad \text{and} \quad f = \frac{\langle f,g\rangle}{\|g\|^2}g + h.$$

Proof Set $h = f - \dfrac{\langle f,g\rangle}{\|g\|^2}g$. Then

$$\langle h, g\rangle = \left\langle f - \frac{\langle f,g\rangle}{\|g\|^2}g, g\right\rangle = \langle f,g\rangle - \frac{\langle f,g\rangle}{\|g\|^2}\langle g,g\rangle = 0,$$

giving the first equation in the conclusion. The second equation in the conclusion follows immediately from the definition of h. ■

The orthogonal decomposition 8.10 is the main ingredient in our proof of the next result, which is one of the most important inequalities in mathematics.

8.11 *Cauchy–Schwarz inequality*

Suppose f and g are elements of an inner product space. Then

$$|\langle f, g\rangle| \leq \|f\| \, \|g\|,$$

with equality if and only if one of f, g is a scalar multiple of the other.

Proof If $g = 0$, then both sides of the desired inequality equal 0. Thus we can assume $g \neq 0$. Consider the orthogonal decomposition

$$f = \frac{\langle f, g\rangle}{\|g\|^2} g + h$$

given by 8.10, where h is orthogonal to g. The Pythagorean Theorem (8.9) implies

$$\begin{aligned} \|f\|^2 &= \left\| \frac{\langle f, g\rangle}{\|g\|^2} g \right\|^2 + \|h\|^2 \\ &= \frac{|\langle f, g\rangle|^2}{\|g\|^2} + \|h\|^2 \end{aligned}$$

8.12
$$\geq \frac{|\langle f, g\rangle|^2}{\|g\|^2}.$$

Multiplying both sides of this inequality by $\|g\|^2$ and then taking square roots gives the desired inequality.

The proof above shows that the Cauchy–Schwarz inequality is an equality if and only if 8.12 is an equality. This happens if and only if $h = 0$. But $h = 0$ if and only if f is a scalar multiple of g (see 8.10). Thus the Cauchy–Schwarz inequality is an equality if and only if f is a scalar multiple of g or g is a scalar multiple of f (or both; the phrasing has been chosen to cover cases in which either f or g equals 0). ■

8.13 Example *Cauchy–Schwarz inequality for* $\mathbf{F}^n$

Applying the Cauchy–Schwarz inequality with the standard inner product on $\mathbf{F}^n$ to $(|a_1|, \ldots, |a_n|)$ and $(|b_1|, \ldots, |b_n|)$ gives the inequality

$$|a_1 b_1| + \cdots + |a_n b_n| \leq \sqrt{|a_1|^2 + \cdots + |a_n|^2} \sqrt{|b_1|^2 + \cdots + |b_n|^2}$$

for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbf{F}^n$.

Thus we have a new and clean proof of Hölder's inequality (7.9) for the special case where μ is counting measure on $\{1, \ldots, n\}$ and $p = p' = 2$.

The inequality in this example was first proved by Cauchy in 1821.

8.14 Example *Cauchy–Schwarz inequality for $L^2(\mu)$*

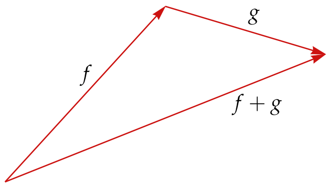
Suppose μ is a measure and $f, g \in L^2(\mu)$. Applying the Cauchy–Schwarz inequality with the standard inner product on $L^2(\mu)$ to $|f|$ and $|g|$ gives the inequality

$$\int |fg|\, d\mu \leq \Big(\int |f|^2\, d\mu\Big)^{1/2} \Big(\int |g|^2\, d\mu\Big)^{1/2}.$$

The inequality above is equivalent to Hölder's inequality (7.9) for the special case where $p = p' = 2$. However, the proof of the inequality above via the Cauchy–Schwarz inequality still depends upon Hölder's inequality to show that the definition of the standard inner product on $L^2(\mu)$ makes sense. See Exercise 18 in this section for a derivation of the inequality above that is truly independent of Hölder's inequality.

In 1859 Viktor Bunyakovsky (1804–1889), who had been Cauchy's student in Paris, first proved integral inequalities like the one above. Similar discoveries by Hermann Schwarz (1843–1921) in 1885 attracted more attention and led to the name of this inequality.

If we think of the norm determined by an inner product as a length, then the triangle inequality has the geometric interpretation that the length of each side of a triangle is less than the sum of the lengths of the other two sides.



8.15 *triangle inequality*

Suppose f and g are elements of an inner product space. Then

$$\|f + g\| \leq \|f\| + \|g\|,$$

with equality if and only if one of f, g is a nonnegative multiple of the other.

Proof We have

$$\begin{aligned}
\|f+g\|^2 &= \langle f+g, f+g\rangle \\
&= \langle f, f\rangle + \langle g, g\rangle + \langle f, g\rangle + \langle g, f\rangle \\
&= \langle f, f\rangle + \langle g, g\rangle + \langle f, g\rangle + \overline{\langle f, g\rangle} \\
&= \|f\|^2 + \|g\|^2 + 2\operatorname{Re}\langle f, g\rangle \\
&\leq \|f\|^2 + \|g\|^2 + 2|\langle f, g\rangle| && (8.16) \\
&\leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\| && (8.17) \\
&= (\|f\| + \|g\|)^2,
\end{aligned}$$

where 8.17 follows from the Cauchy–Schwarz inequality (8.11). Taking square roots of both sides of the inequality above gives the desired inequality.

The proof above shows that the triangle inequality is an equality if and only if we have equality in 8.16 and 8.17. Thus we have equality in the triangle inequality if and only if

8.18 $$\langle f, g\rangle = \|f\|\,\|g\|.$$

If one of f, g is a nonnegative multiple of the other, then 8.18 holds, as you should verify. Conversely, suppose 8.18 holds. Then the condition for equality in the Cauchy–Schwarz inequality (8.11) implies that one of f, g is a scalar multiple of the other. Clearly 8.18 forces the scalar in question to be nonnegative, as desired. ∎

Applying the previous result to the inner product space $L^2(\mu)$, where μ is a measure, gives a new proof of Minkowski's inequality (7.14) for the case $p = 2$.

Now we can prove that what we have been calling a norm on an inner product space is indeed a norm.

8.19 $\|\cdot\|$ *is a norm*

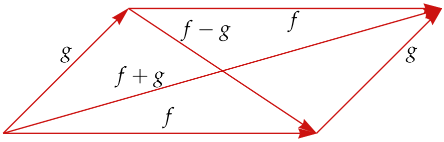
Suppose V is an inner product space and $\|f\|$ is defined as usual by

$$\|f\| = \sqrt{\langle f, f\rangle}$$

for $f \in V$. Then $\|\cdot\|$ is a norm on V.

Proof The definition of an inner product implies that $\|\cdot\|$ satisfies the positive definite requirement for a norm. The homogeneity and triangle inequality requirements for a norm are satisfied because of 8.6 and 8.15. ∎

The next result has the geometric interpretation that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the four sides.



8.20 *parallelogram equality*

Suppose f and g are elements of an inner product space. Then

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

Proof We have

$$\begin{aligned}\|f+g\|^2 + \|f-g\|^2 &= \langle f+g, f+g\rangle + \langle f-g, f-g\rangle \\ &= \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \langle g, f\rangle \\ &\qquad + \|f\|^2 + \|g\|^2 - \langle f, g\rangle - \langle g, f\rangle \\ &= 2\|f\|^2 + 2\|g\|^2,\end{aligned}$$

as desired. ∎

EXERCISES 8A

1 Let V denote the vector space of bounded continuous functions from $\mathbf{R}$ to $\mathbf{F}$. Let $r_1, r_2, \ldots$ be a list of the rational numbers. For $f, g \in V$, define

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \frac{f(r_k)\overline{g(r_k)}}{2^k}.$$

Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.

2 Prove that if $f, g \in L^2(\mu)$, then

$$\|f\|^2 \, \|g\|^2 - |\langle f, g \rangle|^2 = \frac{1}{2} \int \int |f(x)g(y) - g(x)f(y)|^2 \, d\mu(y) \, d\mu(x).$$

3 Suppose f and g are elements of an inner product space and

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

(a) Prove that if $\mathbf{F} = \mathbf{R}$, then f and g are orthogonal.

(b) Give an example to show that if $\mathbf{F} = \mathbf{C}$, then f and g can satisfy the equation above without being orthogonal.

4 Find $a, b \in \mathbf{R}^3$ such that a is a scalar multiple of $(1, 6, 3)$, b is orthogonal to $(1, 6, 3)$, and $(5, 4, -2) = a + b$.

5 Prove that

$$16 \leq (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

for all positive numbers a, b, c, d, with equality if and only if $a = b = c = d$.

6 Prove that the square of the average of each finite list of real numbers containing at least two distinct real numbers is less than the average of the squares of the numbers in that list.

7 Suppose f and g are elements of an inner product space and $\|f\| \leq 1$ and $\|g\| \leq 1$. Prove that

$$\sqrt{1 - \|f\|^2}\sqrt{1 - \|g\|^2} \leq 1 - |\langle f, g \rangle|.$$

8 Suppose a and b are nonzero elements of $\mathbf{R}^2$. Prove that

$$\langle a, b \rangle = \|a\| \, \|b\| \cos \theta,$$

where θ is the angle between a and b (thinking of a as the vector whose initial point is the origin and whose end point is a, and similarly for b).

Hint: Draw the triangle formed by a, b, and $a - b$; then use the law of cosines.

9 The angle between two vectors (thought of as arrows with initial point at the origin) in $\mathbf{R}^2$ or $\mathbf{R}^3$ can be defined geometrically. However, geometry is not as clear in $\mathbf{R}^n$ for $n > 3$. Thus the angle between two nonzero vectors $a, b \in \mathbf{R}^n$ is defined to be

$$\arccos \frac{\langle a, b \rangle}{\|a\| \, \|b\|},$$

where the motivation for this definition comes from the previous exercise. Explain why the Cauchy–Schwarz inequality is needed to show that this definition makes sense.

10 (a) Suppose f and g are elements of a real inner product space. Prove that f and g have the same norm if and only if $f + g$ is orthogonal to $f - g$.

(b) Use part (a) to show that the diagonals of a parallelogram are perpendicular to each other if and only if the parallelogram is a rhombus.

11 Suppose f and g are elements of an inner product space. Prove that $\|f\| = \|g\|$ if and only if $\|sf + tg\| = \|tf + sg\|$ for all $s, t \in \mathbf{R}$.

12 Suppose f and g are elements of an inner product space and $\|f\| = \|g\| = 1$ and $\langle f, g \rangle = 1$. Prove that $f = g$.

13 Suppose f and g are elements of a real inner product space. Prove that

$$\langle f, g \rangle = \frac{\|f + g\|^2 - \|f - g\|^2}{4}.$$

14 Suppose f and g are elements of a complex inner product space. Prove that

$$\langle f, g \rangle = \frac{\|f + g\|^2 - \|f - g\|^2 + \|f + ig\|^2 i - \|f - ig\|^2 i}{4}.$$

15 Suppose f, g, h are elements of an inner product space. Prove that

$$\|h - \tfrac{1}{2}(f + g)\|^2 = \frac{\|h - f\|^2 + \|h - g\|^2}{2} - \frac{\|f - g\|^2}{4}.$$

16 Prove that a norm satisfying the parallelogram equality comes from an inner product. In other words, show that if V is a normed vector space whose norm $\|\cdot\|$ satisfies the parallelogram equality, then there is an inner product $\langle \cdot, \cdot \rangle$ on V such that $\|f\| = \langle f, f \rangle^{1/2}$ for all $f \in V$.

17 Let λ denote Lebesgue measure on $[1, \infty)$.

(a) Prove that if $f \colon [1, \infty) \to [0, \infty)$ is Borel measurable, then

$$\Big(\int_1^\infty f(x) \, d\lambda(x) \Big)^2 \leq \int_1^\infty x^2 \big(f(x) \big)^2 \, d\lambda(x).$$

(b) Describe the set of Borel measurable functions $f \colon [1, \infty) \to [0, \infty)$ such that the inequality in part (a) is an equality.

18 Suppose μ is a measure. For $f, g \in L^2(\mu)$, define $\langle f, g \rangle$ by

$$\langle f, g \rangle = \int f \, \overline{g} \, d\mu.$$

(a) Using the inequality

$$|f(x)\overline{g(x)}| \leq \tfrac{1}{2}\big(|f(x)|^2 + |g(x)|^2\big),$$

verify that the integral above makes sense and the map sending f, g to $\langle f, g \rangle$ defines an inner product on $L^2(\mu)$ (without using Hölder's inequality).

(b) Show that the Cauchy–Schwarz inequality implies that

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2$$

for all $f, g \in L^2(\mu)$ (again, without using Hölder's inequality).

19 Suppose $V_1, \dots, V_m$ are inner product spaces. Show that the equation

$$\langle (f_1, \dots, f_m), (g_1, \dots, g_m) \rangle = \langle f_1, g_1 \rangle + \cdots + \langle f_m, g_m \rangle$$

defines an inner product on $V_1 \times \cdots \times V_m$.
[*Each of the inner product spaces $V_1, \dots, V_m$ may have a different inner product, even though the same inner product notation is used on all these spaces.*]

20 Suppose V is an inner product space. Make $V \times V$ an inner product space as in the exercise above. Prove that the function that takes an ordered pair $(f, g) \in V \times V$ to the inner product $\langle f, g \rangle \in \mathbf{F}$ is a continuous function from $V \times V$ to $\mathbf{F}$.

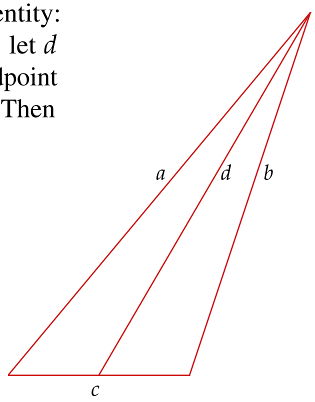
21 Suppose $1 \leq p \leq \infty$.

(a) Show the norm on ℓ^p comes from an inner product if and only if $p = 2$.

(b) Show the norm on $L^p(\mathbf{R})$ comes from an inner product if and only if $p = 2$.

22 Use inner products to prove Apollonius's identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \tfrac{1}{2}c^2 + 2d^2.$$



8B Orthogonality

Orthogonal Projections

The previous section developed inner product spaces following a standard linear algebra approach. Linear algebra focuses mainly on finite-dimensional vector spaces. Many interesting results about infinite-dimensional inner product spaces require an additional hypothesis, which we now introduce.

8.21 Definition ***Hilbert space***

A *Hilbert space* is an inner product space that is a Banach space with the norm determined by the inner product.

8.22 Example *Hilbert spaces*

- Suppose μ is a measure. Then $L^2(\mu)$ with its usual inner product is a Hilbert space (by 7.24).
- As a special case of the first bullet point, if $n \in \mathbf{Z}^+$ then taking μ to be counting measure on $\{1, \ldots, n\}$ shows that $\mathbf{F}^n$ with its usual inner product is a Hilbert space.
- As another special case of the first bullet point, taking μ to be counting measure on $\mathbf{Z}^+$ shows that ℓ^2 with its usual inner product is a Hilbert space.
- Every closed subspace of a Hilbert space is a Hilbert space [by 6.16(b)].

8.23 Example *not Hilbert spaces*

- The inner product space ℓ^1, where $\langle (a_1, a_2, \ldots), (b_1, b_2, \ldots) \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$, is not a Hilbert space because the associated norm is not complete on ℓ^1.
- The inner product space $C([0,1])$ of continuous $\mathbf{F}$-valued functions on the interval $[0,1]$, where $\langle f, g \rangle = \int_0^1 f\overline{g}$, is not a Hilbert space because the associated norm is not complete on $C([0,1])$.

The next definition makes sense in the context of normed vector spaces.

8.24 Definition ***distance from a point to a set***

Suppose U is a nonempty subset of a normed vector space V and $f \in V$. The *distance* from f to U, denoted $\operatorname{distance}(f, U)$, is defined by

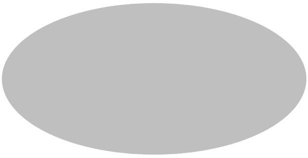
$$\operatorname{distance}(f, U) = \inf\{\|f - g\| : g \in U\}.$$

Notice that $\operatorname{distance}(f, U) = 0$ if and only if $f \in \overline{U}$.

8.25 Definition *convex set*

- A subset of a vector space is called *convex* if the subset contains the line segment connecting each pair of points in it.
- More precisely, suppose V is a vector space and $U \subset V$. Then U is called *convex* if

$$(1-t)f + tg \in U \text{ for all } t \in [0,1] \text{ and all } f, g \in U.$$



Convex subset of $\mathbf{R}^2$.



Nonconvex subset of $\mathbf{R}^2$.

8.26 Example *convex sets*

- Every subspace of a vector space is convex, as you should verify.
- If V is a normed vector space, $f \in V$, and $r > 0$, then the open ball centered at f with radius r is convex, as you should verify.

The next example shows that the distance from an element of a Banach space to a closed subspace is not necessarily attained by some element of the closed subspace. After this example, we will prove that this behavior cannot happen in a Hilbert space.

8.27 Example *no closest element to a closed subspace of a Banach space*

In the Banach space $C([0,1])$ with norm $\|g\| = \sup_{[0,1]} |g|$, let

$$U = \Big\{ g \in C([0,1]) : \int_0^1 g = 0 \text{ and } g(1) = 0 \Big\}.$$

Then U is a closed subspace of $C([0,1])$.

Let $f \in C([0,1])$ be defined by $f(x) = 1 - x$. For $k \in \mathbf{Z}^+$, let

$$g_k(x) = \frac{1}{2} - x + \frac{x^k}{2} + \frac{x-1}{k+1}.$$

Then $g_k \in U$ and $\lim_{k\to\infty} \|f - g_k\| = \frac{1}{2}$, which implies that distance$(f, U) \leq \frac{1}{2}$.

If $g \in U$, then $\int_0^1 (f - g) = \frac{1}{2}$ and $(f - g)(1) = 0$. These conditions imply that $\|f - g\| > \frac{1}{2}$.

Thus distance$(f, U) = \frac{1}{2}$ but there does not exist $g \in U$ such that $\|f - g\| = \frac{1}{2}$.

In the next result, we use for the first time the hypothesis that V is a Hilbert space.

8.28 *distance to a closed convex set is attained in a Hilbert space*

- The distance from an element of a Hilbert space to a nonempty closed convex set is attained by a unique element of the nonempty closed convex set.
- More specifically, suppose V is a Hilbert space, $f \in V$, and U is a nonempty closed convex subset of V. Then there exists a unique $g \in U$ such that

$$\|f - g\| = \text{distance}(f, U).$$

Proof First we prove the existence of an element of U that attains the distance to f. To do this, suppose $g_1, g_2, \dots$ is a sequence of elements of U such that

8.29
$$\lim_{k \to \infty} \|f - g_k\| = \text{distance}(f, U).$$

Then for $j, k \in \mathbf{Z}^+$ we have

$$\begin{aligned}\|g_j - g_k\|^2 &= \|(f - g_k) - (f - g_j)\|^2 \\ &= 2\|f - g_k\|^2 + 2\|f - g_j\|^2 - \|2f - (g_k + g_j)\|^2 \\ &= 2\|f - g_k\|^2 + 2\|f - g_j\|^2 - 4\Big\|f - \frac{g_k + g_j}{2}\Big\|^2 \\ &\le 2\|f - g_k\|^2 + 2\|f - g_j\|^2 - 4\big(\text{distance}(f, U)\big)^2, \qquad (8.30)\end{aligned}$$

where the second equality comes from the parallelogram equality (8.20) and the last line holds because the convexity of U implies that $(g_k + g_j)/2 \in U$. Now the inequality above and 8.29 imply that $g_1, g_2, \dots$ is a Cauchy sequence. Thus there exists $g \in V$ such that

8.31
$$\lim_{k \to \infty} \|g_k - g\| = 0.$$

Because U is a closed subset of V and each $g_k \in U$, we know that $g \in U$. Now 8.29 and 8.31 imply that

$$\|f - g\| = \text{distance}(f, U),$$

which completes the existence proof of the existence part of this result.

To prove the uniqueness part of this result, suppose g and $\tilde{g}$ are elements of U such that

8.32
$$\|f - g\| = \|f - \tilde{g}\| = \text{distance}(f, U).$$

Then

$$\begin{aligned}\|g - \tilde{g}\|^2 &\le 2\|f - g\|^2 + 2\|f - \tilde{g}\|^2 - 4\big(\text{distance}(f, U)\big)^2 \\ &= 0, \qquad (8.33)\end{aligned}$$

where the first line above follows from 8.30 (with g_j replaced by g and g_k replaced by $\tilde{g}$) and the last line above follows from 8.32. Now 8.33 implies that $g = \tilde{g}$, completing the proof of uniqueness. ∎

Example 8.27 showed that the existence part of the previous result can fail in a Banach space. Exercise 13 shows that the uniqueness part can also fail in a Banach space. These observations highlight the advantages of working in a Hilbert space.

8.34 Definition ***orthogonal projection;*** P_U

Suppose U is a nonempty closed convex subset of a Hilbert space V. The *orthogonal projection* of V onto U is the function $P_U\colon V \to V$ defined by setting $P_U(f)$ equal to the unique element of U that is closest to f.

The definition above makes sense because of 8.28. We will often use the notation $P_U f$ instead of $P_U(f)$. To test your understanding of the definition above, make sure that you can show that if U is a nonempty closed convex subset of a Hilbert space V, then

- $P_U f = f$ if and only if $f \in U$;
- $P_U \circ P_U = P_U$.

8.35 Example *orthogonal projection onto closed unit ball*

Suppose U is the closed unit ball $\{g \in V : \|g\| \leq 1\}$ in a Hilbert space V. Then

$$P_U f = \begin{cases} f & \text{if } \|f\| \leq 1, \\ \dfrac{f}{\|f\|} & \text{if } \|f\| > 1, \end{cases}$$

as you should verify.

8.36 Example *orthogonal projection onto a closed subspace*

Suppose U is the closed subspace of ℓ^2 consisting of the elements of ℓ^2 whose even coordinates are all 0:

$$U = \{(a_1, 0, a_3, 0, a_5, 0, \ldots) : \text{each } a_k \in \mathbf{F} \text{ and } \sum_{k=1}^{\infty} |a_{2k-1}|^2 < \infty\}.$$

Then for $b = (b_1, b_2, b_3, b_4, b_5, b_6, \ldots) \in \ell^2$, we have

$$P_U b = (b_1, 0, b_3, 0, b_5, 0, \ldots),$$

as you should verify.

Note that in this example the function P_U is a linear map from ℓ^2 to ℓ^2 (unlike the behavior in Example 8.35).

Also, notice that $b - P_U b = (0, b_2, 0, b_4, 0, b_6, \ldots)$ and thus $b - P_U b$ is orthogonal to every element of U.

The next result shows that the properties stated in the last two paragraphs of the example above hold whenever U is a closed subspace of a Hilbert space.

8.37 *orthogonal projection onto closed subspace*

Suppose U is a closed subspace of a Hilbert space V and $f \in V$. Then

(a) $f - P_U f$ is orthogonal to g for every $g \in U$;

(b) if $h \in U$ and $f - h$ is orthogonal to g for every $g \in U$, then $h = P_U f$;

(c) $P_U \colon V \to V$ is a linear map;

(d) $\|P_U f\| \leq \|f\|$, with equality if and only if $f \in U$.

Proof The figure below illustrates (a). To prove (a), suppose $g \in U$. Then for all $\alpha \in \mathbf{F}$ we have

$$\begin{aligned} \|f - P_U f\|^2 &\leq \|f - P_U f + \alpha g\|^2 \\ &= \langle f - P_U f + \alpha g, f - P_U f + \alpha g \rangle \\ &= \|f - P_U f\|^2 + |\alpha|^2 \|g\|^2 + 2 \operatorname{Re} \overline{\alpha} \langle f - P_U f, g \rangle. \end{aligned}$$

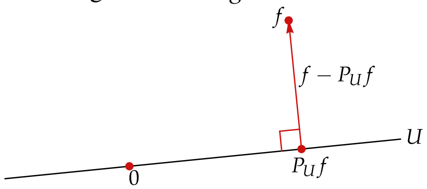
Let $\alpha = -t\langle f - P_U f, g\rangle$ for $t > 0$. A tiny bit of algebra applied to the inequality above implies

$$2|\langle f - P_U f, g\rangle|^2 \leq t|\langle f - P_U f, g\rangle|^2 \|g\|^2$$

for all $t > 0$. Thus $\langle f - P_U f, g\rangle = 0$, completing the proof of (a).

To prove (b), suppose $h \in U$ and $f - h$ is orthogonal to g for every $g \in U$. If $g \in U$, then $h - g \in U$ and hence $f - h$ is orthogonal to $h - g$. Thus

$$\begin{aligned} \|f - h\|^2 &\leq \|f - h\|^2 + \|h - g\|^2 \\ &= \|(f - h) + (h - g)\|^2 \\ &= \|f - g\|^2, \end{aligned}$$



$f - P_U f$ is orthogonal to each element of U.

where the first equality above follows from the Pythagorean Theorem (8.9). Thus

$$\|f - h\| \leq \|f - g\|$$

for all $g \in U$. Hence h is the element of U that minimizes the distance to f, which implies that $h = P_U f$, completing the proof of (b).

To prove (c), suppose $f_1, f_2 \in V$. If $g \in U$, then (a) implies that $\langle f_1 - P_U f_1, g\rangle = \langle f_2 - P_U f_2, g\rangle = 0$, and thus

$$\langle (f_1 + f_2) - (P_U f_1 + P_U f_2), g\rangle = 0.$$

The equation above and (b) now imply that

$$P_U(f_1 + f_2) = P_U f_1 + P_U f_2.$$

The equation above and the equation $P_U(\alpha f) = \alpha P_U f$ for $\alpha \in \mathbf{F}$ (whose proof is left to the reader) show that P_U is a linear map, proving (c).

The proof of (d) is left as an exercise for the reader. ∎

Orthogonal Complements

8.38 Definition *orthogonal complement;* $U^\perp$

Suppose U is a subset of an inner product space V. The *orthogonal complement* of U is denoted by $U^\perp$ and is defined by

$$U^\perp = \{h \in V : \langle g, h\rangle = 0 \text{ for all } g \in U\}.$$

In other words, the orthogonal complement of a subset U of an inner product space V is the set of elements of V that are orthogonal to every element of U.

8.39 Example *orthogonal complement*

Suppose U is the set of elements of ℓ^2 whose even coordinates are all 0:

$$U = \{(a_1, 0, a_3, 0, a_5, 0, \dots) : \text{each } a_k \in \mathbf{F} \text{ and } \sum_{k=1}^{\infty} |a_{2k-1}|^2 < \infty\}.$$

Then $U^\perp$ is the set of elements of ℓ^2 whose odd coordinates are all 0:

$$U^\perp = \{0, a_2, 0, a_4, 0, a_6, \dots) : \text{each } a_k \in \mathbf{F} \text{ and } \sum_{k=1}^{\infty} |a_{2k}|^2 < \infty\},$$

as you should verify.

8.40 ***properties of orthogonal complement***

Suppose U is a subset of an inner product space V. Then

(a) $U^\perp$ is a closed subspace of V;

(b) $U \cap U^\perp \subset \{0\}$;

(c) if $W \subset U$, then $U^\perp \subset W^\perp$;

(d) $\overline{U}^\perp = U^\perp$;

(e) $U \subset (U^\perp)^\perp$.

Proof To prove (a), suppose $h_1, h_2, \dots$ is a sequence in $U^\perp$ that converges to some $h \in V$. If $g \in U$, then

$$|\langle g, h\rangle| = |\langle g, h - h_k\rangle| \le \|g\|\,\|h - h_k\| \quad \text{for each } k \in \mathbf{Z}^+;$$

hence $\langle g, h\rangle = 0$, which implies that $h \in U^\perp$. Thus $U^\perp$ is closed. The proof of (a) is completed by showing that $U^\perp$ is a subspace of V, which is left to the reader.

To prove (b), suppose $g \in U \cap U^\perp$. Then $\langle g, g\rangle = 0$, which implies that $g = 0$, proving (b).

To prove (e), suppose $g \in U$. Thus $\langle g, h\rangle = 0$ for all $h \in U^\perp$, which implies that $g \in (U^\perp)^\perp$. Hence $U \subset (U^\perp)^\perp$, proving (e).

The proofs of (c) and (d) are left to the reader. ■

The results in the rest of this subsection have as a hypothesis that V is a Hilbert space. These results do not hold when V is only an inner product space.

8.41 *orthogonal complement of the orthogonal complement*

Suppose U is a subspace of a Hilbert space V. Then

$$\overline{U} = (U^\perp)^\perp.$$

Proof Applying 8.40(a) to $U^\perp$, we see that $(U^\perp)^\perp$ is a closed subspace of V. Now taking closures of both sides of the inclusion $U \subset (U^\perp)^\perp$ [8.40(e)] shows that $\overline{U} \subset (U^\perp)^\perp$.

To prove the inclusion in the other direction, suppose $f \in (U^\perp)^\perp$. Because $f \in (U^\perp)^\perp$ and $P_{\overline{U}} f \in \overline{U} \subset (U^\perp)^\perp$ (by the previous paragraph), we see that

$$f - P_{\overline{U}} f \in (U^\perp)^\perp.$$

Also,

$$f - P_{\overline{U}} f \in U^\perp$$

by 8.37(a) and 8.40(d). Hence

$$f - P_{\overline{U}} f \in U^\perp \cap (U^\perp)^\perp.$$

Now 8.40(b) (applied to $U^\perp$ in place of U) implies that $f - P_{\overline{U}} f = 0$, which implies that $f \in \overline{U}$. Thus $(U^\perp)^\perp \subset \overline{U}$, completing the proof. ∎

As a special case, the result above implies that if U is a closed subspace of a Hilbert space V, then $U = (U^\perp)^\perp$.

Another special case of the result above is sufficiently useful to deserve stating separately, as we do in the next result.

8.42 *necessary and sufficient condition for a subspace to be dense*

Suppose U is a subspace of a Hilbert space V. Then

$$\overline{U} = V \text{ if and only if } U^\perp = \{0\}.$$

Proof First suppose $\overline{U} = V$. Then using 8.40(d), we have

$$U^\perp = \overline{U}^\perp = V^\perp = \{0\}.$$

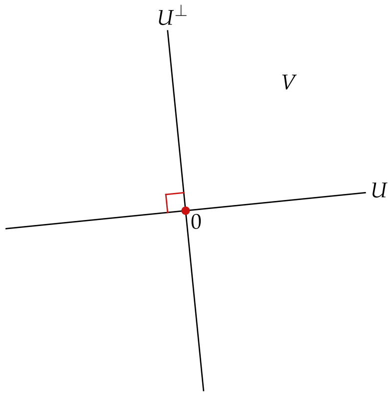
To prove the other direction, now suppose $U^\perp = \{0\}$. Then 8.41 implies that

$$\overline{U} = (U^\perp)^\perp = \{0\}^\perp = V,$$

completing the proof. ∎

The next result states that if U is a closed subspace of a Hilbert space V, then V is the *direct sum* of U and $U^\perp$, often written $V = U \oplus U^\perp$, although we do not need to use this terminology or notation further.

The key point to keep in mind is that the next result shows that the picture here represents what happens in general for a closed subspace U of a Hilbert space V: every element of V can be uniquely written as an element of U plus an element of $U^\perp$.



8.43 *orthogonal decomposition*

Suppose U is a closed subspace of a Hilbert space V. Then every element $f \in V$ can be uniquely written in the form

$$f = g + h,$$

where $g \in U$ and $h \in U^\perp$. Furthermore, $g = P_U f$ and $h = f - P_U f$.

Proof Suppose $f \in V$. Then

$$f = P_U f + (f - P_U f),$$

where $P_U f \in U$ [by definition of $P_U f$ as the element of U that is closest to f] and $f - P_U f \in U^\perp$ [by 8.37(a)]. Thus we have the desired decomposition of f as the sum of an element of U and an element of $U^\perp$.

To prove the uniqueness of this decomposition, suppose

$$f = g_1 + h_1 = g_2 + h_2,$$

where $g_1, g_2 \in U$ and $h_1, h_2 \in U^\perp$. Then $g_1 - g_2 = h_2 - h_1 \in U \cap U^\perp$, which implies that $g_1 = g_2$ and $h_1 = h_2$, as desired. ■

In the next definition, the function I depends upon the vector space V. Thus a notation such as I_V might be more precise. However, the domain of I should always be clear from the context.

8.44 Definition *identity map; I*

Suppose V is a vector space. The *identity map* I is the linear map from V to V defined by $If = f$ for $f \in V$.

The next result highlights the close relationship between orthogonal projections and orthogonal complements.

8.45 *range and null space of orthogonal projections*

Suppose U is a closed subspace of a Hilbert space V. Then

(a) $\operatorname{range} P_U = U$ and $\operatorname{null} P_U = U^\perp$;

(b) $\operatorname{range} P_{U^\perp} = U^\perp$ and $\operatorname{null} P_{U^\perp} = U$;

(c) $P_{U^\perp} = I - P_U$.

Proof The definition of $P_U f$ as the closest point in U to f implies $\operatorname{range} P_U \subset U$. Because $P_U g = g$ for all $g \in U$, we also have $U \subset \operatorname{range} P_U$. Thus $\operatorname{range} P_U = U$.

If $f \in \operatorname{null} P_U$, then $f \in U^\perp$ [by 8.37(a)]. Thus $\operatorname{null} P_U \subset U^\perp$. Conversely, if $f \in U^\perp$, then 8.37(b) (with $h = 0$) implies that $P_U f = 0$; hence $U^\perp \subset \operatorname{null} P_U$. Thus $\operatorname{null} P_U = U^\perp$, completing the proof of (a).

Replace U by $U^\perp$ in (a), getting $\operatorname{range} P_{U^\perp} = U^\perp$ and $\operatorname{null} P_{U^\perp} = (U^\perp)^\perp = U$ (where the last equality comes from 8.41), completing the proof of (b).

Finally, if $f \in U$, then

$$P_{U^\perp} f = 0 = f - P_U f = (I - P_U) f,$$

where the first equality above holds because $\operatorname{null} P_{U^\perp} = U$ [by (b)].

If $f \in U^\perp$, then

$$P_{U^\perp} f = f = f - P_U f = (I - P_U) f,$$

where the second equality above holds because $\operatorname{null} P_U = U^\perp$ [by (a)].

The last two displayed equations show that $P_{U^\perp}$ and $I - P_U$ agree on U and agree on $U^\perp$. Because $P_{U^\perp}$ and $I - P_U$ are both linear maps and because each element of V equals some element of U plus some element of $U^\perp$ (by 8.43), this implies that $P_{U^\perp} = I - P_U$, completing the proof of (c). ∎

8.46 Example $P_{U^\perp} = I - P_U$

Suppose U is the closed subspace of $L^2(\mathbf{R})$ defined by

$$U = \{f \in L^2(\mathbf{R}) : f(x) = 0 \text{ for almost every } x < 0\}.$$

Then, as you should verify,

$$U^\perp = \{f \in L^2(\mathbf{R}) : f(x) = 0 \text{ for almost every } x \geq 0\}.$$

Furthermore, you should also verify that if $f \in L^2(\mathbf{R})$, then

$$P_U f = f\chi_{[0,\,\infty)} \quad \text{and} \quad P_{U^\perp} f = f\chi_{(-\infty,\,0)}.$$

Thus $P_{U^\perp} f = f(1 - \chi_{[0,\,\infty)}) = (I - P_U) f$ and hence $P_{U^\perp} = I - P_U$, as asserted in 8.45(c).

Riesz Representation Theorem

Suppose h is an element of a Hilbert space V. Define $\varphi\colon V \to \mathbf{F}$ by $\varphi(f) = \langle f, h\rangle$ for $f \in V$. The properties of an inner product imply that φ is a linear functional. The Cauchy–Schwarz inequality (8.11) implies that $|\varphi(f)| \leq \|f\|\,\|h\|$ for all $f \in V$, which implies that φ is a bounded linear functional on V. The next result states that every bounded linear functional on V arises in this fashion.

To motivate the proof of the next result, note that if φ is as in the paragraph above, then $\operatorname{null}\varphi = \{h\}^{\perp}$. Thus $h \in (\operatorname{null}\varphi)^{\perp}$ [by 8.40(e)]. Hence in the proof of the next result, to find h we start with an element of $(\operatorname{null}\varphi)^{\perp}$ and then multiply it by a scalar to make everything come out right.

8.47 *Riesz Representation Theorem*

Suppose φ is a bounded linear functional on a Hilbert space V. Then there exists a unique $h \in V$ such that

$$\varphi(f) = \langle f, h\rangle$$

for all $f \in V$. Furthermore, $\|\varphi\| = \|h\|$.

Proof If $\varphi = 0$, take $h = 0$. Thus we can assume $\varphi \neq 0$. Hence $\operatorname{null}\varphi$ is a closed subspace of V not equal to V (see 6.52). The subspace $(\operatorname{null}\varphi)^{\perp}$ is not $\{0\}$ (by 8.42). Thus there exists $g \in (\operatorname{null}\varphi)^{\perp}$ with $\|g\| = 1$. Let

$$h = \overline{\varphi(g)}g.$$

Taking the norm of both sides of the equation above, we get $\|h\| = |\varphi(g)|$. Thus

8.48 $$\varphi(h) = |\varphi(g)|^2 = \|h\|^2.$$

Now suppose $f \in V$. Then

$$\langle f, h\rangle = \Big\langle f - \frac{\varphi(f)}{\|h\|^2}h, h\Big\rangle + \Big\langle \frac{\varphi(f)}{\|h\|^2}h, h\Big\rangle$$

8.49 $$= \Big\langle \frac{\varphi(f)}{\|h\|^2}h, h\Big\rangle$$

$$= \varphi(f),$$

where 8.49 holds because $f - \frac{\varphi(f)}{\|h\|^2}h \in \operatorname{null}\varphi$ (by 8.48) and h is orthogonal to all elements of $\operatorname{null}\varphi$.

We have now proved the existence of $h \in V$ such that $\varphi(f) = \langle f, h\rangle$ for all $f \in V$. To prove uniqueness, suppose $\tilde{h} \in V$ has the same property. Then

$$\langle h - \tilde{h}, h - \tilde{h}\rangle = \langle h - \tilde{h}, h\rangle - \langle h - \tilde{h}, \tilde{h}\rangle = \varphi(h - \tilde{h}) - \varphi(h - \tilde{h}) = 0,$$

which implies that $h = \tilde{h}$, which proves uniqueness.

The Cauchy–Schwarz inequality implies that $|\varphi(f)| = |\langle f, h\rangle| \leq \|f\|\,\|h\|$ for all $f \in V$, which implies that $\|\varphi\| \leq \|h\|$. Because $\varphi(h) = \langle h, h\rangle = \|h\|^2$, we also have $\|\varphi\| \geq \|h\|$. Thus $\|\varphi\| = \|h\|$, completing the proof. ∎

Frigyes Riesz (1880–1956) proved 8.47 in 1907.

Suppose that μ is a measure and $1 < p \leq \infty$. In 7.25 we considered the natural map of $L^{p'}(\mu)$ into $\bigl(L^p(\mu)\bigr)'$, and we showed that this maps preserves norms. In the special case where $p = p' = 2$, the Riesz Representation Theorem (8.47) shows that this map is surjective. In other words, if φ is a bounded linear functional on $L^2(\mu)$, then there exists $h \in L^2(\mu)$ such that

$$\varphi(f) = \int fh\, d\mu$$

for all $f \in L^2(\mu)$ (take h to be the complex conjugate of the function given by 8.47). Hence we can identify the dual of $L^2(\mu)$ with $L^2(\mu)$. In 9.42 we will deal with other values of p. Also see Exercise 25 in this section.

EXERCISES 8B

1 Show that each of the inner product spaces in Example 8.23 is not a Hilbert space.

2 Prove or disprove: The inner product space in Exercise 1 in Section 8A is a Hilbert space.

3 Suppose $V_1, V_2, \dots$ are Hilbert spaces. Let

$$V = \Bigl\{(f_1, f_2, \dots) \in V_1 \times V_2 \times \cdots : \sum_{k=1}^{\infty} \|f_k\|^2 < \infty\Bigr\}.$$

Show that the equation

$$\langle (f_1, f_2, \dots), (g_1, g_2, \dots) \rangle = \sum_{k=1}^{\infty} \langle f_k, g_k \rangle$$

defines an inner product on V that makes V a Hilbert space.
[*Each of the Hilbert spaces $V_1, V_2, \dots$ may have a different inner product, even though the same notation is used for the norm and inner product on all these Hilbert spaces.*]

4 Suppose V is a real Hilbert space. The *complexification* of V is the complex vector space $V_{\mathbf{C}}$ defined by $V_{\mathbf{C}} = V \times V$, but we write a typical element of $V_{\mathbf{C}}$ as $f + ig$ instead of (f, g). Addition and scalar multiplication are defined on $V_{\mathbf{C}}$ by

$$(f_1 + ig_1) + (f_2 + ig_2) = (f_1 + f_2) + i(g_1 + g_2)$$

and

$$(\alpha + \beta i)(f + ig) = (\alpha f - \beta g) + (\alpha g + \beta f)i$$

for $f_1, f_2, f, g_1, g_2, g \in V$ and $\alpha, \beta \in \mathbf{R}$. Show that

$$\langle f_1 + ig_1, f_2 + ig_2 \rangle = \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle + (\langle g_1, f_2 \rangle - \langle f_1, g_2 \rangle)i$$

defines an inner product on $V_{\mathbf{C}}$ that makes $V_{\mathbf{C}}$ into a complex Hilbert space.

5 Prove that if V is a normed vector space, $f \in V$, and $r > 0$, then the open ball $B(f,r)$ centered at f with radius r is convex.

6 (a) Suppose V is an inner product space and B is the open unit ball in V (thus $B = \{f \in V : \|f\| < 1\}$). Prove that if U is a subset of V such that $B \subset U \subset \overline{B}$, then U is convex.

(b) Give an example to show that the result in part (a) can fail if the phrase *inner product space* is replaced by *Banach space*.

7 Suppose V is a normed vector space and U is a closed subset of V. Prove that U is convex if and only if

$$\frac{f+g}{2} \in U \text{ for all } f,g \in U.$$

8 Prove that if U is a convex subset of a normed vector space, then $\overline{U}$ is also convex.

9 Prove that if U is a convex subset of a normed vector space, then the interior of U is also convex.
[*The interior of U is the set $\{f \in U : B(f,r) \subset U$ for some $r > 0\}$.*]

10 Suppose V is a Hilbert space, U is a nonempty closed convex subset of V, and $g \in U$ is the unique element of U with smallest norm (obtained by taking $f = 0$ in 8.28). Prove that

$$\operatorname{Re}\langle g,h\rangle \geq \|g\|^2$$

for all $h \in U$.

11 Suppose V is a Hilbert space. A *closed half-space* of V is a set of the form

$$\{g \in V : \operatorname{Re}\langle g,h\rangle \geq c\}$$

for some $h \in V$ and some $c \in \mathbf{R}$. Prove that every closed convex subset of V is the intersection of all the closed half-spaces that contain it.

12 Give an example of a nonempty closed subset U of the Hilbert space ℓ^2 and $a \in \ell^2$ such that there does not exist $b \in U$ with $\|a-b\| = \operatorname{distance}(a,U)$.
[*By 8.28, U cannot be a convex subset of ℓ^2.*]

13 In the real Banach space $\mathbf{R}^2$ with norm defined by $\|(x,y)\|_\infty = \max\{|x|,|y|\}$, give an example of a closed convex set $U \subset \mathbf{R}^2$ and $z \in \mathbf{R}^2$ such that there exist infinitely many choices of $w \in U$ with $\|z-w\| = \operatorname{distance}(z,U)$.

14 Suppose f and g are elements of an inner product space. Prove that $\langle f,g\rangle = 0$ if and only if

$$\|f\| \leq \|f + \alpha g\|$$

for all $\alpha \in \mathbf{F}$.

15 Suppose U is a closed subspace of a Hilbert space V and $f \in V$. Prove that $\|P_U f\| \leq \|f\|$, with equality if and only if $f \in U$.
[*This exercise asks you to prove 8.37(d).*]

16 Suppose V is a Hilbert space and $P\colon V \to V$ is a linear map such that $P^2 = P$ and $\|Pf\| \le \|f\|$ for every $f \in V$. Prove that there exists a closed subspace U of V such that $P = P_U$.

17 Suppose U is a subspace of a Hilbert space V. Suppose also that W is a Banach space and $S\colon U \to W$ is a bounded linear map. Prove that there exists a bounded linear map $T\colon V \to W$ such that $T|_U = S$ and $\|T\| = \|S\|$.
[*If* $W = \mathbf{F}$*, then this result is just the Hahn–Banach Theorem* (*6.69*) *for Hilbert spaces. The result here is stronger because it allows* W *to be an arbitrary Banach space instead of requiring* W *to be* $\mathbf{F}$*. Also, the proof in this Hilbert space context does not require use of Zorn's Lemma or the Axiom of Choice.*]

18 Suppose U and W are subspaces of a Hilbert space V. Prove that $\overline{U} = \overline{W}$ if and only if $U^\perp = W^\perp$.

19 Suppose U and W are closed subspaces of a Hilbert space. Prove that $P_U P_W = 0$ if and only if $\langle f, g\rangle = 0$ for all $f \in U$ and all $g \in W$.

20 Verify the assertions in Example 8.46.

21 Show that every inner product space is a subspace of some Hilbert space.

Hint: See Exercise 13 in Section 6C.

22 Prove that if T is a bounded linear operator on a Hilbert space V and the dimension of range T is 1, then there exist $g, h \in V$ such that

$$Tf = \langle f, g\rangle h$$

for all $f \in V$.

23 (a) Give an example of a Banach space V and a bounded linear functional φ on V such that $|\varphi(f)| < \|\varphi\|\,\|f\|$ for all $f \in V \setminus \{0\}$.

(b) Show there does not exist an example in part (a) where V is a Hilbert space.

24 (a) Suppose φ and ψ are bounded linear functionals on a Hilbert space V such that $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$. Prove that one of φ, ψ is a scalar multiple of the other.

(b) Give an example to show that part (a) can fail if the hypothesis that V is a Hilbert space is replaced by the hypothesis that V is a Banach space.

25 (a) Suppose that μ is a finite measure, $1 \le p \le 2$, and φ is a bounded linear functional on $L^p(\mu)$. Prove that there exists $h \in L^{p'}(\mu)$ such that $\varphi(f) = \int fh\,d\mu$ for every $f \in L^p(\mu)$.

(b) Same as part (a), but with the hypothesis that μ is a finite measure replaced by the hypothesis that μ is a measure.

[*See 7.25, which along with this exercise shows that we can identify the dual of* $L^p(u)$ *with* $L^{p'}(\mu)$ *for* $1 < p \le 2$*. See 9.42 for an extension to all* $p \in (1, \infty)$*.*]

26 Prove that if V is a infinite-dimensional Hilbert space, then the Banach space $\mathcal{B}(V)$ is nonseparable.

8C Orthonormal Bases

Bessel's Inequality

Recall that a *family* $\{e_k\}_{k\in\Gamma}$ in a set V is a function e from a set Γ to V, with the value of the function e at $k \in \Gamma$ denoted by e_k (see 6.53).

8.50 Definition *orthonormal family*

A family $\{e_k\}_{k\in\Gamma}$ in an inner product space is called an *orthonormal family* if

$$\langle e_j, e_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k \end{cases}$$

for all $j, k \in \Gamma$.

In other words, a family $\{e_k\}_{k\in\Gamma}$ is an orthonormal family if e_j and e_k are orthogonal for all distinct $j, k \in \Gamma$ and $\|e_k\| = 1$ for all $k \in \Gamma$.

8.51 Example *orthonormal families*

- For $k \in \mathbf{Z}^+$, let e_k be the element of ℓ^2 all of whose coordinates are 0 except for the kth coordinate, which is 1:

  $$e_k = (0, \ldots, 0, 1, 0, \ldots).$$

  Then $\{e_k\}_{k\in\mathbf{Z}^+}$ is an orthonormal family in ℓ^2. In this case, our family is a sequence; thus we can call $\{e_k\}_{k\in\mathbf{Z}^+}$ an *orthonormal sequence*.

- More generally, suppose Γ is a nonempty set. The Hilbert space $L^2(\mu)$, where μ is counting measure on Γ, is often denoted by $\ell^2(\Gamma)$. For $k \in \Gamma$, define a function $e_k \colon \Gamma \to \mathbf{F}$ by

  $$e_k(j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

  Then $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in $\ell^2(\Gamma)$.

- For $k \in \mathbf{Z}$, define $e_k \colon (-\pi, \pi] \to \mathbf{R}$ by

  $$e_k(t) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(kt) & \text{if } k > 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0, \\ \frac{1}{\sqrt{\pi}} \cos(kt) & \text{if } k < 0. \end{cases}$$

  Then $\{e_k\}_{k\in\mathbf{Z}}$ is an orthonormal family in $L^2\big((-\pi, \pi]\big)$, as you should verify (see Exercise 1 for useful formulas that will help with this verification).

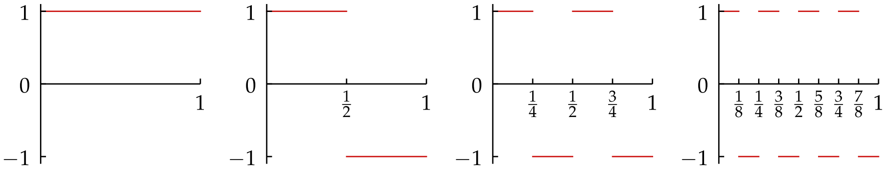
  This orthonormal family $\{e_k\}_{k\in\mathbf{Z}}$ leads to the classical theory of Fourier series, as we will see in more depth in Chapter 11.

- For k a nonnegative integer, define $e_k \colon [0,1) \to \mathbf{F}$ by

$$e_k(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right) \text{ for some odd integer } n, \\ -1 & \text{if } x \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right) \text{ for some even integer } n. \end{cases}$$

The figure below shows the graphs of e_0, e_1, e_2, and e_3. The pattern of these graphs should convince you that $\{e_k\}_{k\in\{0,1,\ldots\}}$ is an orthonormal family in $L^2([0,1))$.

This orthonormal family was invented by Hans Rademacher (1892–1969).



The graph of e_0. *The graph of e_1.* *The graph of e_2.* *The graph of e_3.*

- Now we modify the example in the previous bullet point by translating the functions in the previous bullet point by arbitrary integers. Specifically, for k a nonnegative integer and $m \in \mathbf{Z}$, define $e_{k,m} \colon \mathbf{R} \to \mathbf{F}$ by

$$e_{k,m}(x) = \begin{cases} 1 & \text{if } x \in \left[m + \frac{n-1}{2^k}, m + \frac{n}{2^k}\right) \text{ for some odd integer } n \in [1, 2^k], \\ -1 & \text{if } x \in \left[m + \frac{n-1}{2^k}, m + \frac{n}{2^k}\right) \text{ for some even integer } n \in [1, 2^k], \\ 0 & \text{if } x \notin [m, m+1). \end{cases}$$

Then $\{e_{k,m}\}_{(k,m)\in\{0,1,\ldots\}\times\mathbf{Z}}$ is an orthonormal family in $L^2(\mathbf{R})$.

This example illustrates the usefulness of considering families that are not sequences. Although $\{0,1,\ldots\}\times\mathbf{Z}$ is a countable set and hence we could rewrite $\{e_{k,m}\}_{(k,m)\in\{0,1,\ldots\}\times\mathbf{Z}}$ as a sequence, doing so would be awkward and would be less clean than the $e_{k,m}$ notation.

The next result gives our first indication of why orthonormal families are so useful.

8.52 *finite orthonormal families*

Suppose Ω is a finite set and $\{e_j\}_{j\in\Omega}$ is an orthonormal family in an inner product space. Then

$$\Big\| \sum_{j\in\Omega} \alpha_j e_j \Big\|^2 = \sum_{j\in\Omega} |\alpha_j|^2$$

for every family $\{\alpha_j\}_{j\in\Omega}$ in $\mathbf{F}$.

Proof Suppose $\{\alpha_j\}_{j\in\Omega}$ is a family in $\mathbf{F}$. Standard properties of inner products show that

$$\begin{aligned}\Big\|\sum_{j\in\Omega}\alpha_j e_j\Big\|^2 &= \Big\langle \sum_{j\in\Omega}\alpha_j e_j, \sum_{k\in\Omega}\alpha_k e_k\Big\rangle\\ &= \sum_{j,k\in\Omega}\alpha_j\overline{\alpha_k}\langle e_j, e_k\rangle\\ &= \sum_{j\in\Omega}|\alpha_j|^2,\end{aligned}$$

as desired. ■

Suppose Ω is a finite set and $\{e_j\}_{j\in\Omega}$ is an orthonormal family in an inner product space. The result above implies that if $\sum_{j\in\Omega}\alpha_j e_j = 0$, then $\alpha_j = 0$ for every $j\in\Omega$.

Linear algebra, and algebra more generally, deals with sums of only finitely many terms. However, in analysis we often want to sum infinitely many terms. For example, earlier we defined the infinite sum of a sequence $g_1, g_2, \dots$ in a normed vector space to be the limit as $n\to\infty$ of the partial sums $\sum_{k=1}^n g_k$ if that limit exists (see 6.40).

The next definition captures a more powerful method of dealing with infinite sums. The sum defined below is called an *unordered sum* because the set Γ is not assumed to come with any ordering. A finite unordered sum is defined in the obvious way.

8.53 Definition ***unordered sum;*** $\sum_{k\in\Gamma} f_k$

Suppose $\{f_k\}_{k\in\Gamma}$ is a family in a normed vector space V. The *unordered sum* $\sum_{k\in\Gamma} f_k$ is said to *converge* if there exists $g\in V$ such that for every $\varepsilon>0$, there exists a finite subset Ω of Γ such that

$$\Big\|g-\sum_{j\in\Omega'} f_j\Big\| < \varepsilon$$

for all finite sets Ω' with $\Omega\subset\Omega'\subset\Gamma$. If this happens, we set $\sum_{k\in\Gamma} f_k = g$. If there is no such $g\in V$, then $\sum_{k\in\Gamma} f_k$ is left undefined.

Exercises at the end of this section ask you to develop basic properties of unordered sums, including the following:

- Suppose $\{a_k\}_{k\in\Gamma}$ is a family in $\mathbf{R}$ and $a_k\geq 0$ for each $k\in\Gamma$. Then the unordered sum $\sum_{k\in\Gamma} a_k$ converges if and only if

$$\sup\Big\{\sum_{j\in\Omega} a_j : \Omega \text{ is a finite subset of } \Gamma\Big\} < \infty.$$

  Furthermore, if $\sum_{k\in\Gamma} a_k$ converges then it equals the supremum above. If $\sum_{k\in\Gamma} a_k$ does not converge, then the supremum above is ∞ and we write $\sum_{k\in\Gamma} a_k = \infty$ (this notation should be used only when $a_k\geq 0$ for each $k\in\Gamma$).

- Suppose $\{a_k\}_{k\in\Gamma}$ is a family in $\mathbf{R}$. Then the unordered sum $\sum_{k\in\Gamma} a_k$ converges if and only if $\sum_{k\in\Gamma}|a_k| < \infty$. Thus convergence of an unordered summation in $\mathbf{R}$ is the same as absolute convergence. As we are about to see, the situation in more general Hilbert spaces is quite different.

Now we can extend 8.52 to infinite sums.

8.54 *linear combinations of an orthonormal family*

Suppose $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in a Hilbert space V. Suppose $\{\alpha_k\}_{k\in\Gamma}$ is a family in $\mathbf{F}$. Then

(a) $$\text{the unordered sum } \sum_{k\in\Gamma} \alpha_k e_k \text{ converges} \iff \sum_{k\in\Gamma} |\alpha_k|^2 < \infty.$$

Furthermore, if $\sum_{k\in\Gamma} \alpha_k e_k$ converges, then

(b) $$\Big\|\sum_{k\in\Gamma} \alpha_k e_k\Big\|^2 = \sum_{k\in\Gamma} |\alpha_k|^2.$$

Proof First suppose $\sum_{k\in\Gamma} \alpha_k e_k$ converges, with $\sum_{k\in\Gamma} \alpha_k e_k = g$. Suppose $\varepsilon > 0$. Then there exists a finite set $\Omega \subset \Gamma$ such that

$$\Big\|g - \sum_{j\in\Omega'} \alpha_j e_j\Big\| < \varepsilon$$

for all finite sets Ω' with $\Omega \subset \Omega' \subset \Gamma$. If Ω' is a finite set with $\Omega \subset \Omega' \subset \Gamma$, then the inequality above implies that

$$\|g\| - \varepsilon < \Big\|\sum_{j\in\Omega'} \alpha_j e_j\Big\| < \|g\| + \varepsilon,$$

which (using 8.52) implies that

$$\|g\| - \varepsilon < \Big(\sum_{j\in\Omega'} |\alpha_j|^2\Big)^{1/2} < \|g\| + \varepsilon.$$

Thus $\|g\| = \big(\sum_{k\in\Gamma}|\alpha_k|^2\big)^{1/2}$, completing the proof of one direction of (a) and the proof of (b).

To prove the other direction of (a), now suppose $\sum_{k\in\Gamma}|\alpha_k|^2 < \infty$. Thus there exists an increasing sequence $\Omega_1 \subset \Omega_2 \subset \cdots$ of finite subsets of Γ such that for each $m \in \mathbf{Z}^+$,

8.55 $$\sum_{j\in\Omega'\setminus\Omega_m} |\alpha_j|^2 < \frac{1}{m^2}$$

for every finite set Ω' such that $\Omega_m \subset \Omega' \subset \Gamma$. For each $m \in \mathbf{Z}^+$, let

$$g_m = \sum_{j\in\Omega_m} \alpha_j e_j.$$

If $n > m$, then 8.52 implies that

$$\|g_n - g_m\|^2 = \sum_{j \in \Omega_n \setminus \Omega_m} |\alpha_j|^2 < \frac{1}{m^2}.$$

Thus $g_1, g_2, \dots$ is a Cauchy sequence and hence converges to some element g of V.

Temporarily fixing $m \in \mathbf{Z}^+$ and taking the limit of the equation above as $n \to \infty$, we see that

$$\|g - g_m\| \leq \frac{1}{m}.$$

To show that $\sum_{k \in \Gamma} \alpha_k e_k = g$, suppose $\varepsilon > 0$. Let $m \in \mathbf{Z}^+$ be such that $\frac{2}{m} < \varepsilon$. Suppose Ω' is a finite set with $\Omega_m \subset \Omega' \subset \Gamma$. Then

$$\begin{aligned}
\Big\|g - \sum_{j \in \Omega'} \alpha_j e_j\Big\| &\leq \|g - g_m\| + \Big\|g_m - \sum_{j \in \Omega'} \alpha_j e_j\Big\| \\
&\leq \frac{1}{m} + \Big\|\sum_{j \in \Omega' \setminus \Omega_m} \alpha_j e_j\Big\| \\
&= \frac{1}{m} + \Big(\sum_{j \in \Omega' \setminus \Omega_m} |\alpha_j|^2\Big)^{1/2} \\
&< \varepsilon,
\end{aligned}$$

where the third line comes from 8.52 and the last line comes from 8.55. Thus $\sum_{k \in \Gamma} \alpha_k e_k = g$, completing the proof. ∎

8.56 **Example** *a convergent unordered sum need not converge absolutely*

Suppose $\{e_k\}_{k \in \mathbf{Z}^+}$ is the orthogonal family in ℓ^2 defined by setting e_k equal to the sequence that is 0 everywhere except for a 1 in the kth slot. Then by 8.54, the unordered sum

$$\sum_{k \in \mathbf{Z}^+} \tfrac{1}{k} e_k$$

converges in ℓ^2 (because $\sum_{k \in \mathbf{Z}^+} \frac{1}{k^2} < \infty$) even though $\sum_{k \in \mathbf{Z}^+} \|\frac{1}{k} e_k\| = \infty$. Note that $\sum_{k \in \mathbf{Z}^+} \frac{1}{k} e_k = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in \ell^2$.

Now we prove an important inequality.

8.57 ***Bessel's inequality***

Suppose $\{e_k\}_{k \in \Gamma}$ is an orthonormal family in an inner product space V and $f \in V$. Then

$$\sum_{k \in \Gamma} |\langle f, e_k \rangle|^2 \leq \|f\|^2.$$

Bessel's inequality is named in honor of Friedrich Bessel (1784–1846), who discovered this inequality in 1828 in the special case of the trigonometric orthonormal family given by the third bullet point in Example 8.51.

Proof Suppose Ω is a finite subset of Γ. Then

$$f = \sum_{j\in\Omega} \langle f, e_j \rangle e_j + \Big(f - \sum_{j\in\Omega} \langle f, e_j \rangle e_j\Big),$$

where the first sum above is orthogonal to the term in parentheses above (as you should verify).

Applying the Pythagorean Theorem (8.9) to the equation above gives

$$\begin{aligned}\|f\|^2 &= \Big\|\sum_{j\in\Omega} \langle f, e_j \rangle e_j\Big\|^2 + \Big\|f - \sum_{j\in\Omega} \langle f, e_j \rangle e_j\Big\|^2 \\ &\geq \Big\|\sum_{j\in\Omega} \langle f, e_j \rangle e_j\Big\|^2 \\ &= \sum_{j\in\Omega} |\langle f, e_j \rangle|^2,\end{aligned}$$

where the last equality follows from 8.52. Because the inequality above holds for every finite set $\Omega \subset \Gamma$, we conclude that $\|f\|^2 \geq \sum_{k\in\Gamma} |\langle f, e_k \rangle|^2$, as desired. ■

Recall that the span of a family $\{e_k\}_{k\in\Gamma}$ in a vector space is the set of finite sums of the form

$$\sum_{j\in\Omega} \alpha_j e_j,$$

where Ω is a finite subset of Γ and $\{\alpha_j\}_{j\in\Omega}$ is a family in $\mathbf{F}$ (see 6.54). Bessel's inequality now allows us to prove the following beautiful result showing that the closure of the span of an orthonormal family is a set of infinite sums.

8.58 *closure of the span of an orthonormal family*

Suppose $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in a Hilbert space V. Then

(a) $$\overline{\operatorname{span}\{e_k\}_{k\in\Gamma}} = \Big\{\sum_{k\in\Gamma} \alpha_k e_k : \{\alpha_k\}_{k\in\Gamma} \text{ is a family in } \mathbf{F} \text{ and } \sum_{k\in\Gamma} |\alpha_k|^2 < \infty\Big\}.$$

Furthermore,

(b) $$f = \sum_{k\in\Gamma} \langle f, e_k \rangle e_k$$

for every $f \in \overline{\operatorname{span}\{e_k\}_{k\in\Gamma}}$.

Proof The right side of (a) above makes sense because of 8.54(a). Furthermore, the right side of (a) above is a subspace of V because $\ell^2(\Gamma)$ [which equals $\mathcal{L}^2(\mu)$, where μ is counting measure on Γ] is closed under addition and scalar multiplication by 7.5.

Suppose first $\{\alpha_k\}_{k\in\Gamma}$ is a family in $\mathbf{F}$ and $\sum_{k\in\Gamma}|\alpha_k|^2 < \infty$. Let $\varepsilon > 0$. Then there is a finite subset Ω of Γ such that

$$\sum_{j\in\Gamma\setminus\Omega} |\alpha_j|^2 < \varepsilon^2.$$

The inequality above and 8.54(b) imply that

$$\Big\|\sum_{k\in\Gamma} \alpha_k e_k - \sum_{j\in\Omega} \alpha_j e_j\Big\| < \varepsilon.$$

The definition of the closure (see 6.7) now implies that $\sum_{k\in\Gamma} \alpha_k e_k \in \overline{\operatorname{span}\,\{e_k\}_{k\in\Gamma}}$, showing that the right side of (a) is contained in the left side of (a).

To prove the inclusion in the other direction, now suppose $f \in \overline{\operatorname{span}\,\{e_k\}_{k\in\Gamma}}$. Let

8.59 $$g = \sum_{k\in\Gamma} \langle f, e_k\rangle e_k,$$

where the sum above converges by Bessel's inequality (8.57) and by 8.54(a). The direction of the inclusion that we just proved implies that $g \in \overline{\operatorname{span}\,\{e_k\}_{k\in\Gamma}}$. Thus

8.60 $$g - f \in \overline{\operatorname{span}\,\{e_k\}_{k\in\Gamma}}.$$

Equation 8.59 implies that $\langle g, e_j\rangle = \langle f, e_j\rangle$ for each $j \in \Gamma$, as you should verify (which will require using the Cauchy–Schwarz inequality if done rigorously). Hence

$$\langle g - f, e_k\rangle = 0 \quad \text{for every } k \in \Gamma.$$

This implies that

$$g - f \in \big(\operatorname{span}\{e_j\}_{j\in\Gamma}\big)^{\perp} = \big(\overline{\operatorname{span}\{e_j\}_{j\in\Gamma}}\big)^{\perp},$$

where the equality above comes from 8.40(d). Now 8.60 and the inclusion above imply that $f = g$ [see 8.40(b)], which along with 8.59 implies that f is in the right side of (a), completing the proof of (a).

The equations $f = g$ and 8.59 also imply (b). ■

Parseval's Identity

Note that 8.52 implies that every orthonormal family in an inner product space is linearly independent (see 6.54 to review the definition of linearly independent and basis). Linear algebra deals mainly with finite-dimensional vector spaces, but infinite-dimensional vector spaces frequently appear in analysis. The notion of a basis is not so useful when doing analysis with infinite-dimensional vector spaces because the definition of span does not take advantage of the possibility of summing an infinite number of elements.

However, 8.58 tells us that taking the closure of the span of an orthonormal family can capture the sum of infinitely many elements. Thus we make the following definition.

8.61 Definition ***orthonormal basis***

An orthonormal family $\{e_k\}_{k\in\Gamma}$ in a Hilbert space V is called an *orthonormal basis* of V if

$$\overline{\operatorname{span}\{e_k\}_{k\in\Gamma}} = V.$$

In addition to requiring orthonormality (which implies linear independence), the definition above differs from the definition of a basis by considering the closure of the span rather than the span. An important point to keep in mind is that despite the terminology, an orthonormal basis is not necessarily a basis in the sense of 6.54. In fact, if Γ is an infinite set and $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of V, then $\{e_k\}_{k\in\Gamma}$ is not a basis of V (see Exercise 9).

8.62 Example *orthonormal bases*

- For $n \in \mathbf{Z}^+$ and $k \in \{1,\dots,n\}$, let e_k be the element of $\mathbf{F}^n$ all of whose coordinates are 0 except the kth coordinate, which is 1:
$$e_k = (0,\dots,0,1,0,\dots,0).$$
Then $\{e_k\}_{k\in\{1,\dots,n\}}$ is an orthonormal basis of $\mathbf{F}^n$.
- Let $e_1 = \big(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\big)$, $e_2 = \big(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\big)$, and $e_3 = \big(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\big)$. Then $\{e_k\}_{k\in\{1,2,3\}}$ is an orthonormal basis of $\mathbf{F}^3$, as you should verify.
- The first three bullet points in 8.51 are examples of orthonormal families that are orthonormal bases. The exercises ask you to verify that we have an orthonormal basis in the first and second bullet points of 8.51. For the third bullet point (trigonometric functions), see Exercise 11 in Section 10D or see Chapter 11.

The next result shows why orthonormal bases are so useful—a Hilbert space with an orthonormal basis $\{e_k\}_{k\in\Gamma}$ behaves like $\ell^2(\Gamma)$.

8.63 ***Parseval's identity***

Suppose $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of a Hilbert space V and $f,g \in V$. Then

(a) $f = \displaystyle\sum_{k\in\Gamma} \langle f,e_k\rangle e_k$;

(b) $\langle f,g\rangle = \displaystyle\sum_{k\in\Gamma} \langle f,e_k\rangle \overline{\langle g,e_k\rangle}$;

(c) $\|f\|^2 = \displaystyle\sum_{k\in\Gamma} |\langle f,e_k\rangle|^2$.

Proof The equation in (a) follows immediately from 8.58(b) and the definition of an orthonormal basis.

To prove (b), note that

$$\begin{aligned}\langle f, g\rangle &= \Big\langle \sum_{k\in\Gamma}\langle f, e_k\rangle e_k, g\Big\rangle\\ &= \sum_{k\in\Gamma}\langle f, e_k\rangle\langle e_k, g\rangle\\ &= \sum_{k\in\Gamma}\langle f, e_k\rangle\overline{\langle g, e_k\rangle},\end{aligned}$$

Equation (c) is called Parseval's identity *in honor of Marc-Antoine Parseval (1755–1836), who discovered a special case in 1799.*

where the first equation follows from (a) and the second equation follows from the definition of an unordered sum and the Cauchy–Schwarz inequality.

Equation (c) follows from setting $g = f$ in (b). An alternative proof: equation (c) follows from 8.54(b) and the equation $f = \sum_{k\in\Gamma}\langle f, e_k\rangle e_k$ from (a). ■

Gram–Schmidt Process and Existence of Orthonormal Bases

8.64 Definition ***separable***

A normed vector space is called *separable* if it has a countable subset whose closure equals the whole space.

8.65 Example *separable normed vector spaces*

- Suppose $n \in \mathbf{Z}^+$. Then $\mathbf{F}^n$ with the usual Hilbert space norm is separable because the closure of the countable set

  $$\{(c_1, \dots, c_n) \in \mathbf{F}^n : \text{each } c_j \text{ is rational}\}$$

  equals $\mathbf{F}^n$ (in case $\mathbf{F} = \mathbf{C}$: to say that a complex number is rational in this context means that both the real and imaginary parts of the complex number are rational numbers in the usual sense).

- The Hilbert space ℓ^2 is separable because the closure of the countable set

  $$\bigcup_{n=1}^{\infty}\{(c_1, \dots, c_n, 0, 0, \dots) \in \ell^2 : \text{ each } c_j \text{ is rational}\}$$

  is ℓ^2.

- The Hilbert spaces $L^2([0,1])$ and $L^2(\mathbf{R})$ are separable, as Exercise 13 asks you to verify [hint: consider finite linear combinations with rational coefficients of functions of the form $\chi_{(c,d)}$, where c and d are rational numbers].

A moment's thought about the definition of closure (see 6.7) shows that a normed vector space V is separable if and only if there exists a countable subset C of V such that every open ball in V contains at least one element of C.

8.66 **Example** *nonseparable normed vector spaces*

- Suppose Γ is an uncountable set. Then the Hilbert space $\ell^2(\Gamma)$ is not separable. To see this, note that $\|\chi_{\{j\}} - \chi_{\{k\}}\| = \sqrt{2}$ for all $j, k \in \Gamma$ with $j \neq k$. Hence

$$\left\{B\left(\chi_{\{k\}}, \tfrac{\sqrt{2}}{2}\right) : k \in \Gamma\right\}$$

is an uncountable collection of disjoint open balls in $\ell^2(\Gamma)$; no countable set can have at least one element in each of these balls.

- The Banach space $L^\infty([0,1])$ is not separable. Here $\|\chi_{[0,s]} - \chi_{[0,t]}\| = 1$ for all $s, t \in [0,1]$ with $s \neq t$. Thus

$$\left\{B\left(\chi_{[0,\,t]}, \tfrac{1}{2}\right) : t \in [0,1]\right\}$$

is an uncountable collection of disjoint open balls in $L^\infty([0,1])$.

We present two proofs of the existence of orthonormal bases of Hilbert spaces. The first proof works only for separable Hilbert spaces, but it gives a useful algorithm, called the *Gram–Schmidt process*, for constructing orthonormal sequences. The second proof works for all Hilbert spaces, but it uses a result that depends upon the Axiom of Choice.

Which proof should you read? In practice, the Hilbert spaces you will encounter will almost certainly be separable. Thus the first proof suffices, and it has the additional benefit of introducing you to a widely used algorithm. The second proof uses an entirely different approach and has the advantage of applying to separable and nonseparable Hilbert spaces. For maximum learning, read both proofs!

8.67 ***existence of orthonormal bases for separable Hilbert spaces***

Every separable Hilbert space has an orthonormal basis.

Proof Suppose V is a separable Hilbert space and $\{f_1, f_2, \ldots\}$ is a countable subset of V whose closure equals V. We will inductively define an orthonormal sequence $\{e_k\}_{k\in\mathbf{Z}^+}$ such that

8.68 $$\operatorname{span}\{f_1, \ldots, f_n\} \subset \operatorname{span}\{e_1, \ldots, e_n\}$$

for each $n \in \mathbf{Z}^+$. This will imply that $\overline{\operatorname{span}\{e_k\}_{k\in\mathbf{Z}^+}} = V$, which will mean that $\{e_k\}_{k\in\mathbf{Z}^+}$ is an orthonormal basis of V.

To get started with the induction, set $e_1 = f_1/\|f_1\|$ (we can assume that $f_1 \neq 0$).

Now suppose $n \in \mathbf{Z}^+$ and $e_1, \dots, e_n$ have been chosen so that $\{e_k\}_{k\in\{1,\dots,n\}}$ is an orthonormal family in V and 8.68 holds. If $f_k \in \operatorname{span}\{e_1, \dots, e_n\}$ for every $k \in \mathbf{Z}^+$, then $\{e_k\}_{k\in\{1,\dots,n\}}$ is an orthonormal basis of V (completing the proof) and the process should be stopped. Otherwise, let m be the smallest positive integer such that

8.69 $$f_m \notin \operatorname{span}\{e_1, \dots, e_n\}.$$

Define e_{n+1} by

8.70 $$e_{n+1} = \frac{f_m - \langle f_m, e_1\rangle e_1 - \cdots - \langle f_m, e_n\rangle e_n}{\|f_m - \langle f_m, e_1\rangle e_1 - \cdots - \langle f_m, e_n\rangle e_n\|}.$$

Jørgen Gram (1850–1916) and Erhard Schmidt (1876–1959) popularized this process that constructs orthonormal sequences.

Clearly $\|e_{n+1}\| = 1$ (8.69 guarantees there is no division by 0). If $k \in \{1, \dots, n\}$, then the equation above implies that $\langle e_{n+1}, e_k\rangle = 0$. Thus $\{e_k\}_{k\in\{1,\dots,n+1\}}$ is an orthonormal family in V. Also, 8.68 and the choice of m as the smallest positive integer satisfying 8.69 imply that

$$\operatorname{span}\{f_1, \dots, f_{n+1}\} \subset \operatorname{span}\{e_1, \dots, e_{n+1}\},$$

completing the induction and completing the proof. ■

Before considering nonseparable Hilbert spaces, we take a short detour to illustrate how the Gram–Schmidt process used in the previous proof can be used to find closest elements to subspaces. We begin with a result connecting the orthogonal projection onto a closed subspace with an orthonormal basis of that subspace.

8.71 *orthogonal projection in terms of an orthonormal basis*

Suppose that U is a closed subspace of a Hilbert space V and $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of U. Then

$$P_U f = \sum_{k\in\Gamma} \langle f, e_k\rangle e_k$$

for all $f \in V$.

Proof Let $f \in V$. If $k \in \Gamma$, then

8.72 $$\langle f, e_k\rangle = \langle f - P_U f, e_k\rangle + \langle P_U f, e_k\rangle = \langle P_U f, e_k\rangle,$$

where the last equality follows from 8.37(a). Now

$$P_U f = \sum_{k\in\Gamma} \langle P_U f, e_k\rangle e_k = \sum_{k\in\Gamma} \langle f, e_k\rangle e_k,$$

where the first equality follows from Parseval's identity [8.63(a)] as applied to U and its orthonormal basis $\{e_k\}_{k\in\Gamma}$, and the second equality follows from 8.72. ■

8.73 Example *best approximation*

Find the polynomial g of degree at most 10 that minimizes

$$\int_{-1}^{1} \left| \sqrt{|x|} - g(x) \right|^2 dx.$$

Solution We will work in the real Hilbert space $L^2([-1,1])$ with the usual inner product $\langle g, h \rangle = \int_{-1}^{1} gh$. For $k \in \{0, 1, \dots, 10\}$, let $f_k \in L^2([-1,1])$ be defined by $f_k(x) = x^k$. Let U be the subspace of $L^2([-1,1])$ defined by

$$U = \operatorname{span}\{f_k\}_{k \in \{0,\dots,10\}}.$$

Apply the Gram–Schmidt process from the proof of 8.67 to $\{f_k\}_{k\in\{0,\dots,10\}}$, producing an orthonormal basis $\{e_k\}_{k\in\{0,\dots,10\}}$ of U, which is a closed subspace of $L^2([-1,1])$ (see Exercise 8). The point here is that $\{e_k\}_{k\in\{0,\dots,10\}}$ can be computed explicitly and exactly by using 8.70 and evaluating some integrals (using software that can do exact rational arithmetic will make the process easier), getting $e_0(x) = 1/\sqrt{2}$, $e_1(x) = \sqrt{6}x/2, \dots$ up to

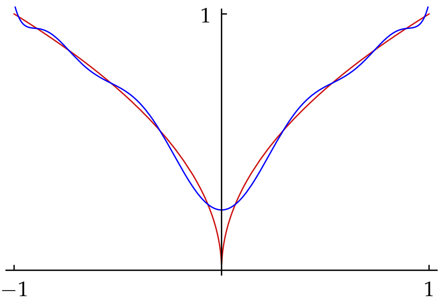
$$e_{10}(x) = \frac{\sqrt{42}}{512}(-63 + 3465x^2 - 30030x^4 + 90090x^6 - 109395x^8 + 46189x^{10}).$$

Define $f \in L^2([-1,1])$ by $f(x) = \sqrt{|x|}$. Because U is the subspace of $L^2([-1,1])$ consisting of polynomials of degree at most 10 and $P_U f$ equals the element of U closest to f (see 8.34), the formula in 8.71 tells us that the solution g to our minimization problem is given by the formula

$$g = \sum_{k=0}^{10} \langle f, e_k \rangle e_k.$$

Using the explicit expressions for $e_0, \dots, e_{10}$ and again evaluating some integrals, this gives

$$g(x) = \frac{693 + 15015x^2 - 64350x^4 + 139230x^6 - 138567x^8 + 51051x^{10}}{2944}.$$



The figure here shows the graph of $f(x) = \sqrt{|x|}$ (red) and the graph of its closest polynomial g (blue) of degree at most 10; here *closest* means as measured in the norm of $L^2([-1,1])$.

The approximation of f by g is pretty good, especially considering that f is not differentiable at 0 and thus a Taylor series expansion for f does not make sense.

Recall that a subset Γ of a set V can be thought of as a family in V by considering $\{e_f\}_{f \in \Gamma}$, where $e_f = f$. With this convention, a subset Γ of an inner product space V is an *orthonormal subset* of V if $\|f\| = 1$ for all $f \in \Gamma$ and $\langle f, g \rangle = 0$ for all $f, g \in \Gamma$ with $f \neq g$.

The next result characterizes the orthonormal bases as the maximal elements among the collection of orthonormal subsets of a Hilbert space. Recall that a set $\Gamma \in \mathcal{A}$ in a collection of subsets of a set V is a maximal element of $\mathcal{A}$ if there does not exist $\Gamma' \in \mathcal{A}$ such that $\Gamma \subsetneq \Gamma'$ (see 6.55).

8.74 *orthonormal bases as maximal elements*

Suppose V is a Hilbert space, $\mathcal{A}$ is the collection of all orthonormal subsets of V, and Γ is an orthonormal subset of V. Then Γ is an orthonormal basis of V if and only if Γ is a maximal element of $\mathcal{A}$.

Proof First suppose Γ is an orthonormal basis of V. Parseval's identity [8.63(a)] implies that the only element of V that is orthogonal to every element of Γ is 0. Thus there does not exist an orthonormal subset of V that strictly contains Γ. In other words, Γ is a maximal element of $\mathcal{A}$.

To prove the other direction, suppose now that Γ is a maximal element of $\mathcal{A}$. Let U denote the span of Γ. Then

$$U^{\perp} = \{0\}$$

because if f is a nonzero element of $U^{\perp}$, then $\Gamma \cup \{f / \|f\|\}$ is an orthonormal subset of V that strictly contains Γ. Hence $\overline{U} = V$ (by 8.42), which implies that Γ is an orthonormal basis of V. ■

Now we are ready to prove that every Hilbert space has an orthonormal basis. Before reading the next proof, you may want to review the definition of a chain (6.58), which is a collection of sets such that for each pair of sets in the collection, one of them is contained in the other. You should also review Zorn's Lemma (6.60), which gives a way to show that a collection of sets contains a maximal element.

8.75 *existence of orthonormal bases for all Hilbert spaces*

Every Hilbert space has an orthonormal basis.

Proof Suppose V is a Hilbert space. Let $\mathcal{A}$ be the collection of all orthonormal subsets of V. Suppose $\mathcal{C} \subset \mathcal{A}$ is a chain. Let L be the union of all the sets in $\mathcal{C}$. If $f \in L$, then $\|f\| = 1$ because f is an element of some orthonormal subset of V that is contained in $\mathcal{C}$.

If $f, g \in L$ with $f \neq g$, then there exist orthonormal subsets Ω and Γ in $\mathcal{C}$ such that $f \in \Omega$ and $g \in \Gamma$. Because $\mathcal{C}$ is a chain, either $\Omega \subset \Gamma$ or $\Gamma \subset \Omega$. Either way, there is an orthonormal subset of V that contains both f and g. Thus $\langle f, g \rangle = 0$.

We have shown that L is an orthonormal subset of V; in other words, $L \in \mathcal{A}$. Thus Zorn's Lemma (6.60) implies that $\mathcal{A}$ has a maximal element. Now 8.74 implies that V has an orthonormal basis. ■

Riesz Representation Theorem, Revisited

Now that we know that every Hilbert space has an orthonormal basis, we can give a completely different proof of the Riesz Representation Theorem (8.47) than the proof we gave earlier.

Note that the new proof below of the Riesz Representation Theorem gives the formula 8.77 for h in terms of an orthonormal basis. One interesting feature of this formula is that h is uniquely determined by φ and thus h does not depend upon the choice of an orthonormal basis. Hence despite its appearance, the right side of 8.77 is independent of the choice of an orthonormal basis.

8.76 *Riesz Representation Theorem*

Suppose φ is a bounded linear functional on a Hilbert space V and $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of V. Let

8.77
$$h = \sum_{k\in\Gamma} \overline{\varphi(e_k)}e_k.$$

Then

8.78
$$\varphi(f) = \langle f, h\rangle$$

for all $f \in V$. Furthermore, $\|\varphi\| = \bigl(\sum_{k\in\Gamma}|\varphi(e_k)|^2\bigr)^{1/2}$.

Proof First we must show that the sum defining h makes sense. To do this, suppose Ω is a finite subset of Γ. Then

$$\sum_{j\in\Omega}|\varphi(e_j)|^2 = \varphi\Bigl(\sum_{j\in\Omega}\overline{\varphi(e_j)}e_j\Bigr) \le \|\varphi\|\,\Bigl\|\sum_{j\in\Omega}\overline{\varphi(e_j)}e_j\Bigr\| = \|\varphi\|\Bigl(\sum_{j\in\Omega}|\varphi(e_j)|^2\Bigr)^{1/2},$$

where the last equality follows from 8.52. Dividing by $\bigl(\sum_{j\in\Omega}|\varphi(e_j)|^2\bigr)^{1/2}$ gives

$$\Bigl(\sum_{j\in\Omega}|\varphi(e_j)|^2\Bigr)^{1/2} \le \|\varphi\|.$$

Because the inequality above holds for every finite subset Ω of Γ, we conclude that

$$\sum_{k\in\Gamma}|\varphi(e_k)|^2 \le \|\varphi\|^2.$$

Thus the sum defining h makes sense (by 8.54) in equation 8.77.

Now 8.77 shows that $\langle h, e_j\rangle = \overline{\varphi(e_j)}$ for each $j \in \Gamma$. Thus if $f \in V$ then

$$\varphi(f) = \varphi\Bigl(\sum_{k\in\Gamma}\langle f, e_k\rangle e_k\Bigr) = \sum_{k\in\Gamma}\langle f, e_k\rangle\varphi(e_k) = \sum_{k\in\Gamma}\langle f, e_k\rangle\overline{\langle h, e_k\rangle} = \langle f, h\rangle,$$

where the first and last equalities follow from 8.63 and the second equality follows from the boundedness/continuity of φ. Thus 8.78 holds.

Finally, the Cauchy–Schwarz inequality, equation 8.78, and the equation $\varphi(h) = \langle h, h\rangle$ show that $\|\varphi\| = \|h\| = \bigl(\sum_{k\in\Gamma}|\varphi(e_k)|^2\bigr)^{1/2}$. ■

EXERCISES 8C

1 Verify that the family $\{e_k\}_{k\in\mathbf{Z}}$ as defined in the third bullet point of Example 8.51 is an orthonormal family in $L^2\big((-\pi,\pi]\big)$. The following formulas should help:

$$(\sin x)(\cos y) = \frac{\sin(x+y)+\sin(x-y)}{2},$$
$$(\sin x)(\sin y) = \frac{\cos(x-y)-\cos(x+y)}{2},$$
$$(\cos x)(\cos y) = \frac{\cos(x+y)+\cos(x-y)}{2}.$$

2 Suppose $\{a_k\}_{k\in\Gamma}$ is a family in $\mathbf{R}$ and $a_k \geq 0$ for each $k \in \Gamma$. Prove the unordered sum $\sum_{k\in\Gamma} a_k$ converges if and only if

$$\sup\Big\{\sum_{j\in\Omega} a_j : \Omega \text{ is a finite subset of } \Gamma\Big\} < \infty.$$

Furthermore, prove that if $\sum_{k\in\Gamma} a_k$ converges then it equals the supremum above.

3 Suppose $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in an inner product space V. Prove that if $f \in V$, then $\{k \in \Gamma : \langle f, e_k\rangle \neq 0\}$ is a countable set.

4 Suppose $\{f_k\}_{k\in\Gamma}$ and $\{g_k\}_{k\in\Gamma}$ are families in a normed vector space such that $\sum_{k\in\Gamma} f_k$ and $\sum_{k\in\Gamma} g_k$ converge. Prove that $\sum_{k\in\Gamma}(f_k+g_k)$ converges and

$$\sum_{k\in\Gamma}(f_k+g_k) = \sum_{k\in\Gamma} f_k + \sum_{k\in\Gamma} g_k.$$

5 Suppose $\{f_k\}_{k\in\Gamma}$ is a family in a normed vector space such that $\sum_{k\in\Gamma} f_k$ converges. Prove that if $c \in \mathbf{F}$, then $\sum_{k\in\Gamma}(cf_k)$ converges and

$$\sum_{k\in\Gamma}(cf_k) = c\sum_{k\in\Gamma} f_k.$$

6 Suppose $\{a_k\}_{k\in\Gamma}$ is a family in $\mathbf{R}$. Prove that the unordered sum $\sum_{k\in\Gamma} a_k$ converges if and only if $\sum_{k\in\Gamma}|a_k| < \infty$.

7 Suppose $\{f_k\}_{k\in\mathbf{Z}^+}$ is a family in a normed vector space. Prove that the unordered sum $\sum_{k\in\mathbf{Z}^+} f_k$ converges if and only if the usual ordered sum $\sum_{k=1}^{\infty} f_{p(k)}$ converges for every injective function $p\colon \mathbf{Z}^+ \to \mathbf{Z}^+$.

8 Explain why 8.58 implies that if Γ is a finite set and $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in a Hilbert space V, then $\operatorname{span}\{e_k\}_{k\in\Gamma}$ is a closed subspace of V.

9 Suppose V is an infinite-dimensional Hilbert space. Prove that there does not exist a basis of V that is an orthonormal family.

10 (a) Show that the orthonormal family given in the first bullet point of Example 8.51 is an orthonormal basis of ℓ^2.

(b) Show that the orthonormal family given in the second bullet point of Example 8.51 is an orthonormal basis of $\ell^2(\Gamma)$.

(c) Show that the orthonormal family given in the fourth bullet point of Example 8.51 is not an orthonormal basis of $L^2\big([0,1)\big)$.

(d) Show that the orthonormal family given in the fifth bullet point of Example 8.51 is not an orthonormal basis of $L^2(\mathbf{R})$.

11 Suppose μ is a σ-finite measure on $(X,\mathcal{S})$ and ν is a σ-finite measure on $(Y,\mathcal{T})$. Suppose also that $\{e_j\}_{j\in\Omega}$ is an orthonormal basis of $L^2(\mu)$ and $\{f_k\}_{k\in\Gamma}$ is an orthonormal basis of $L^2(\nu)$. For $j\in\Omega$ and $k\in\Gamma$, define $g_{j,k}\colon X\times Y\to\mathbf{F}$ by

$$g_{j,k}(x,y)=e_j(x)f_k(y).$$

Prove that $\{g_{j,k}\}_{j\in\Omega,\,k\in\Gamma}$ is an orthonormal basis of $L^2(\mu\times\nu)$.

12 Prove the converse of Parseval's identity. More specifically, prove that if $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in a Hilbert space V and

$$\|f\|^2=\sum_{k\in\Gamma}|\langle f,e_k\rangle|^2$$

for every $f\in V$, then $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of V.

13 (a) Show that the Hilbert space $L^2([0,1])$ is separable.

(b) Show that the Hilbert space $L^2(\mathbf{R})$ is separable.

(c) Show that the Banach space ℓ^∞ is not separable.

14 Prove that every subspace of a separable normed vector space is separable.

15 Suppose V is an infinite-dimensional Hilbert space. Prove that there does not exist a translation invariant measure on the Borel subsets of V that assigns positive but finite measure to each open ball in V.
[*A subset of V is called a* Borel set *if it is in the smallest σ-algebra containing all the open subsets of V. A measure μ on the Borel subsets of V is called* translation invariant *if $\mu(f+E)=\mu(E)$ for every $f\in V$ and every Borel set E of V.*]

16 Find the polynomial g of degree at most 4 that minimizes $\displaystyle\int_0^1 \left|x^5-g(x)\right|^2\,dx$.

17 Prove that each orthonormal family in a Hilbert space can be extended to an orthonormal basis of the Hilbert space. Specifically, suppose $\{e_j\}_{j\in\Omega}$ is an orthonormal family in a Hilbert space V. Prove that there exists a set Γ containing Ω and an orthonormal basis $\{f_k\}_{k\in\Gamma}$ of V such that $f_j=e_j$ for every $j\in\Omega$.

18 Prove that every vector space has a basis.

19 Find the polynomial g of degree at most 4 such that

$$f\left(\tfrac{1}{2}\right) = \int_0^1 fg$$

for every polynomial f of degree at most 4.

Exercises 20–25 are for readers familiar with analytic functions.

20 Suppose G is a nonempty open subset of $\mathbf{C}$. The *Bergman space* $L_a^2(G)$ is defined to be the set of analytic functions $f\colon G \to \mathbf{C}$ such that

$$\int_G |f|^2 \, d\lambda_2 < \infty,$$

where λ_2 is the usual Lebesgue measure on $\mathbf{R}^2$, which is identified with $\mathbf{C}$. For $f, h \in L_a^2(G)$, define $\langle f, h\rangle$ to be $\int_G f\,\overline{h}\, d\lambda_2$.

(a) Show that $L_a^2(G)$ is a Hilbert space.

(b) Show that if $w \in G$, then $f \mapsto f(w)$ is a bounded linear functional on $L_a^2(G)$.

21 Let $\mathbf{D}$ denote the open unit disk in $\mathbf{C}$; thus

$$\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}.$$

(a) Find an orthonormal basis of $L_a^2(\mathbf{D})$.

(b) Suppose $f \in L_a^2(\mathbf{D})$ has Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

for $z \in \mathbf{D}$. Find a formula for $\|f\|$ in terms of $a_0, a_1, a_2, \dots$.

(c) Suppose $w \in \mathbf{D}$. By the previous exercise and the Riesz Representation Theorem (8.47 and 8.76), there exists $\Gamma_w \in L_a^2(\mathbf{D})$ such that

$$f(w) = \langle f, \Gamma_w\rangle \text{ for all } f \in L_a^2(\mathbf{D}).$$

Find an explicit formula for Γ_w.

22 Suppose G is the annulus defined by

$$G = \{z \in \mathbf{C} : 1 < |z| < 2\}.$$

(a) Find an orthonormal basis of $L_a^2(G)$.

(b) Suppose $f \in L_a^2(G)$ has Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

for $z \in G$. Find a formula for $\|f\|$ in terms of $\dots, a_{-1}, a_0, a_1, \dots$.

23 Prove that if $f \in L^2_a(\mathbf{D} \setminus \{0\})$, then f has a removable singularity at 0 (meaning that f can be extended to a function that is analytic on $\mathbf{D}$).

24 The *Dirichlet space* $\mathcal{D}$ is defined to be the set of analytic functions $f\colon \mathbf{D} \to \mathbf{C}$ such that

$$\int_{\mathbf{D}} |f'|^2 \, d\lambda_2 < \infty.$$

For $f, g \in \mathcal{D}$, define $\langle f, g \rangle$ to be $f(0)\overline{g(0)} + \int_{\mathbf{D}} f' \, \overline{g'} \, d\lambda_2$.

(a) Show that $\mathcal{D}$ is a Hilbert space.

(b) Show that if $w \in \mathbf{D}$, then $f \mapsto f(w)$ is a bounded linear functional on $\mathcal{D}$.

(c) Find an orthonormal basis of $\mathcal{D}$.

(d) Suppose $f \in \mathcal{D}$ has Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

for $z \in \mathbf{D}$. Find a formula for $\|f\|$ in terms of $a_0, a_1, a_2, \dots$.

(e) Suppose $w \in \mathbf{D}$. Find an explicit formula for $\Gamma_w \in \mathcal{D}$ such that

$$f(w) = \langle f, \Gamma_w \rangle \text{ for all } f \in \mathcal{D}.$$

25 (a) Prove that the Dirichlet space $\mathcal{D}$ is contained in the Bergman space $L^2_a(\mathbf{D})$.

(b) Prove that there exists a function $f \in L^2_a(\mathbf{D})$ such that f is uniformly continuous on $\mathbf{D}$ and $f \notin \mathcal{D}$.

Chapter 9
Real and Complex Measures

A measure is a countably additive function from a σ-algebra to $[0, \infty]$. In this chapter, we consider countably additive functions from a σ-algebra to either **R** or **C**. The first section of this chapter shows that these functions, called real measures or complex measures, form an interesting Banach space with an appropriate norm.

The second section of this chapter focuses on decomposition theorems that help us understand real and complex measures. These results will lead to a proof that the dual space of $L^p(\mu)$ can be identified with $L^{p'}(\mu)$.



Dome in the main building of the University of Vienna, where Johann Radon (1887–1956) was a student and then later a faculty member. The Radon–Nikodym Theorem, which will be proved in this chapter using Hilbert space techniques, provides information analogous to differentiation for measures.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_9

9A Total Variation

Properties of Real and Complex Measures

Recall that a measurable space is a pair $(X, \mathcal{S})$, where $\mathcal{S}$ is a σ-algebra on X. Recall also that a measure on $(X, \mathcal{S})$ is a countably additive function from $\mathcal{S}$ to $[0, \infty]$ that takes $\emptyset$ to 0. Countably additive functions that take values in $\mathbf{R}$ or $\mathbf{C}$ give us new objects called real measures or complex measures.

9.1 Definition ***real and complex measures***

Suppose $(X, \mathcal{S})$ is measurable space.

- A function $\nu \colon \mathcal{S} \to \mathbf{F}$ is called *countably additive* if

$$\nu\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \sum_{k=1}^{\infty} \nu(E_k)$$

  for every disjoint sequence $E_1, E_2, \ldots$ of sets in $\mathcal{S}$.

- A *real measure* on $(X, \mathcal{S})$ is a countably additive function $\nu \colon \mathcal{S} \to \mathbf{R}$.
- A *complex measure* on $(X, \mathcal{S})$ is a countably additive function $\nu \colon \mathcal{S} \to \mathbf{C}$.

The terminology nonnegative measure *would be more appropriate than* positive measure *because the function* $\mu \colon \mathcal{S} \to \mathbf{F}$ *defined by* $\mu(E) = 0$ *for every* $E \in \mathcal{S}$ *is a positive measure. However, we will stick with tradition and use the phrase* positive measure.

The word *measure* can be ambiguous in the mathematical literature. The most common use of the word *measure* is as we defined it in Chapter 2 (see 2.54). However, some mathematicians use the word *measure* to include what are here called real and complex measures; they then use the phrase *positive measure* to refer to what we defined as a measure in 2.54. To help relieve this ambiguity, in this chapter we usually use the phrase (*positive*) *measure* to refer to measures as defined in 2.54. Putting *positive* in parentheses helps reinforce the idea that it is optional while distinguishing such measures from real and complex measures.

9.2 Example *real and complex measures*

- Let λ denote Lebesgue measure on $[-1, 1]$. Define ν on the Borel subsets of $[-1, 1]$ by

$$\nu(E) = \lambda\bigl(E \cap [0, 1]\bigr) - \lambda\bigl(E \cap [-1, 0)\bigr).$$

  Then ν is a real measure.

- If μ_1 and μ_2 are finite (positive) measures, then $\mu_1 - \mu_2$ is a real measure and $\alpha_1\mu_1 + \alpha_2\mu_2$ is a complex measure for all $\alpha_1, \alpha_2 \in \mathbf{C}$.
- If ν is a complex measure, then $\operatorname{Re}\nu$ and $\operatorname{Im}\nu$ are real measures.

Note that every real measure is a complex measure. Note also that by definition, ∞ is not an allowable value for a real or complex measure. Thus a (positive) measure μ on $(X, \mathcal{S})$ is a real measure if and only if $\mu(X) < \infty$.

Some authors use the terminology *signed measure* instead of *real measure*; some authors allow a real measure to take on the value ∞ or $-\infty$ (but not both, because the expression $\infty - \infty$ must be avoided). However, real measures as defined here serve us better because we need to avoid $\pm\infty$ when considering the Banach space of real or complex measures on a measurable space (see 9.18).

For (positive) measures, we had to make $\mu(\emptyset) = 0$ part of the definition to avoid the function μ that assigns ∞ to all sets, including the empty set. But ∞ is not an allowable value for real or complex measures. Thus $\nu(\emptyset) = 0$ is a consequence of our definition rather than part of the definition, as shown in the next result.

9.3 *absolute convergence for a disjoint union*

Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. Then

(a) $\nu(\emptyset) = 0$;

(b) $\displaystyle\sum_{k=1}^{\infty} |\nu(E_k)| < \infty$ for every disjoint sequence $E_1, E_2, \ldots$ of sets in $\mathcal{S}$.

Proof To prove (a), note that $\emptyset, \emptyset, \ldots$ is a disjoint sequence of sets in $\mathcal{S}$ whose union equals $\emptyset$. Thus

$$\nu(\emptyset) = \sum_{k=1}^{\infty} \nu(\emptyset).$$

The right side of the equation above makes sense as an element of $\mathbf{R}$ or $\mathbf{C}$ only when $\nu(\emptyset) = 0$, which proves (a).

To prove (b), suppose $E_1, E_2, \ldots$ is a disjoint sequence of sets in $\mathcal{S}$. First suppose ν is a real measure. Thus

$$\nu\Bigl(\bigcup_{\{k : \nu(E_k) > 0\}} E_k\Bigr) = \sum_{\{k : \nu(E_k) > 0\}} \nu(E_k) = \sum_{\{k : \nu(E_k) > 0\}} |\nu(E_k)|$$

and

$$-\nu\Bigl(\bigcup_{\{k : \nu(E_k) < 0\}} E_k\Bigr) = -\sum_{\{k : \nu(E_k) < 0\}} \nu(E_k) = \sum_{\{k : \nu(E_k) < 0\}} |\nu(E_k)|.$$

Because $\nu(E) \in \mathbf{R}$ for every $E \in \mathcal{S}$, the right side of the last two displayed equations is finite. Thus $\sum_{k=1}^{\infty} |\nu(E_k)| < \infty$, as desired.

Now consider the case where ν is a complex measure. Then

$$\sum_{k=1}^{\infty} |\nu(E_k)| \leq \sum_{k=1}^{\infty} \bigl(|(\operatorname{Re} \nu)(E_k)| + |(\operatorname{Im} \nu)(E_k)|\bigr) < \infty,$$

where the last inequality follows from applying the result for real measures to the real measures $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$. ∎

The next definition provides an important class of examples of real and complex measures.

9.4 *measure determined by an $\mathcal{L}^1$-function*

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $h \in \mathcal{L}^1(\mu)$. Define $\nu\colon \mathcal{S} \to \mathbf{F}$ by

$$\nu(E) = \int_E h \, d\mu.$$

Then ν is a real measure on $(X, \mathcal{S})$ if $\mathbf{F} = \mathbf{R}$ and is a complex measure on $(X, \mathcal{S})$ if $\mathbf{F} = \mathbf{C}$.

Proof Suppose $E_1, E_2, \dots$ is a disjoint sequence of sets in $\mathcal{S}$. Then

9.5 $$\nu\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \int \Bigl(\sum_{k=1}^{\infty} \chi_{E_k}(x) h(x)\Bigr)\, d\mu(x) = \sum_{k=1}^{\infty} \int \chi_{E_k} h \, d\mu = \sum_{k=1}^{\infty} \nu(E_k),$$

where the first equality holds because the sets $E_1, E_2, \dots$ are disjoint and the second equality follows from the inequality

$$\Bigl|\sum_{k=1}^{m} \chi_{E_k}(x) h(x)\Bigr| \le |h(x)|,$$

which along with the assumption that $h \in \mathcal{L}^1(\mu)$ allows us to interchange the integral and limit of the partial sums by the Dominated Convergence Theorem (3.31).

The countable additivity shown in 9.5 means ν is a real or complex measure. ■

The next definition simply gives a notation for the measure defined in the previous result. In the notation that we are about to define, the symbol d has no separate meaning—it functions to separate h and μ.

9.6 Definition $h \, d\mu$

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $h \in \mathcal{L}^1(\mu)$. Then $h \, d\mu$ is the real or complex measure on $(X, \mathcal{S})$ defined by

$$(h \, d\mu)(E) = \int_E h \, d\mu.$$

Note that if a function $h \in \mathcal{L}^1(\mu)$ takes values in $[0, \infty)$, then $h \, d\mu$ is a finite (positive) measure.

The next result shows some basic properties of complex measures. No proofs are given because the proofs are the same as the proofs of the corresponding results for (positive) measures. Specifically, see the proofs of 2.57, 2.61, 2.59, and 2.60. Because complex measures cannot take on the value ∞, we do not need to worry about hypotheses of finite measure that are required of the (positive) measure versions of all but part (c).

9.7 *properties of complex measures*

Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. Then

(a) $\nu(E \setminus D) = \nu(E) - \nu(D)$ for all $D, E \in \mathcal{S}$ with $D \subset E$;

(b) $\nu(D \cup E) = \nu(D) + \nu(E) - \nu(D \cap E)$ for all $D, E \in \mathcal{S}$;

(c) $\nu\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \lim_{k \to \infty} \nu(E_k)$

for all increasing sequences $E_1 \subset E_2 \subset \cdots$ of sets in $\mathcal{S}$;

(d) $\nu\Bigl(\bigcap_{k=1}^{\infty} E_k\Bigr) = \lim_{k \to \infty} \nu(E_k)$

for all decreasing sequences $E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$.

Total Variation Measure

We use the terminology *total variation measure* below even though we have not yet shown that the object being defined is a measure. Soon we will justify this terminology (see 9.11).

9.8 Definition *total variation measure*

Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. The *total variation measure* is the function $|\nu| : \mathcal{S} \to [0, \infty]$ defined by

$$|\nu|(E) = \sup\{|\nu(E_1)| + \cdots + |\nu(E_n)| : n \in \mathbf{Z}^+ \text{ and } E_1, \ldots, E_n \text{ are disjoint sets in } \mathcal{S} \text{ such that } E_1 \cup \cdots \cup E_n \subset E\}.$$

To start getting familiar with the definition above, you should verify that if ν is a complex measure on $(X, \mathcal{S})$ and $E \in \mathcal{S}$, then

- $|\nu(E)| \leq |\nu|(E)$;
- $|\nu|(E) = \nu(E)$ if ν is a finite (positive) measure;
- $|\nu|(E) = 0$ if and only if $\nu(A) = 0$ for every $A \in \mathcal{S}$ such that $A \subset E$.

The next result states that for real measures, we can consider only $n = 2$ in the definition of the total variation measure.

9.9 *total variation measure of a real measure*

Suppose ν is a real measure on a measurable space $(X, \mathcal{S})$ and $E \in \mathcal{S}$. Then

$$|\nu|(E) = \sup\{|\nu(A)| + |\nu(B)| : A, B \text{ are disjoint sets in } \mathcal{S} \text{ and } A \cup B \subset E\}.$$

Proof Suppose that $n \in \mathbf{Z}^+$ and $E_1, \dots, E_n$ are disjoint sets in $\mathcal{S}$ such that $E_1 \cup \cdots \cup E_n \subset E$. Let

$$A = \bigcup_{\{k : \nu(E_k) > 0\}} E_k \quad \text{and} \quad B = \bigcup_{\{k : \nu(E_k) < 0\}} E_k.$$

Then A, B are disjoint sets in $\mathcal{S}$ and $A \cup B \subset E$. Furthermore,

$$|\nu(A)| + |\nu(B)| = |\nu(E_1)| + \cdots + |\nu(E_n)|.$$

Thus in the supremum that defines $|\nu|(E)$, we can take $n = 2$. ■

The next result could be rephrased as stating that if $h \in \mathcal{L}^1(\mu)$, then the total variation measure of the measure $h\,d\mu$ is the measure $|h|\,d\mu$. In the statement below, the notation $d\nu = h\,d\mu$ means the same as $\nu = h\,d\mu$; the notation $d\nu$ is commonly used when considering expressions involving measures of the form $h\,d\mu$.

9.10 *total variation measure of* $h\,d\mu$

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$, $h \in \mathcal{L}^1(\mu)$, and $d\nu = h\,d\mu$. Then

$$|\nu|(E) = \int_E |h|\,d\mu$$

for every $E \in \mathcal{S}$.

Proof Suppose that $E \in S$. If $E_1, \dots, E_n$ is a disjoint sequence in $\mathcal{S}$ such that $E_1 \cup \cdots \cup E_n \subset E$, then

$$\sum_{k=1}^{n} |\nu(E_k)| = \sum_{k=1}^{n} \left| \int_{E_k} h\,d\mu \right| \leq \sum_{k=1}^{n} \int_{E_k} |h|\,d\mu \leq \int_E |h|\,d\mu.$$

The inequality above implies that $|\nu|(E) \leq \int_E |h|\,d\mu$.

To prove the inequality in the other direction, first suppose $\mathbf{F} = \mathbf{R}$; thus h is a real-valued function and ν is a real measure. Let

$$A = \{x \in E : h(x) > 0\} \quad \text{and} \quad B = \{x \in E : h(x) < 0\}.$$

Then A and B are disjoint sets in $\mathcal{S}$ and $A \cup B \subset E$. We have

$$|\nu(A)| + |\nu(B)| = \int_A h\,d\mu - \int_B h\,d\mu = \int_E |h|\,d\mu.$$

Thus $|\nu|(E) \geq \int_E |h|\,d\mu$, completing the proof in the case $\mathbf{F} = \mathbf{R}$.

Now suppose $\mathbf{F} = \mathbf{C}$; thus ν is a complex measure. Let $\varepsilon > 0$. There exists a simple function $g \in \mathcal{L}^1(\mu)$ such that $\|g - h\|_1 < \varepsilon$ (by 3.44). There exist disjoint sets $E_1, \dots, E_n \in \mathcal{S}$ and $c_1, \dots, c_n \in \mathbf{C}$ such that $E_1 \cup \cdots \cup E_n \subset E$ and

$$g|_E = \sum_{k=1}^{n} c_k \chi_{E_k}.$$

Now

$$\begin{aligned}\sum_{k=1}^{n}|\nu(E_k)| &= \sum_{k=1}^{n}\Big|\int_{E_k} h\,d\mu\Big|\\ &\geq \sum_{k=1}^{n}\Big|\int_{E_k} g\,d\mu\Big| - \sum_{k=1}^{n}\Big|\int_{E_k}(g-h)\,d\mu\Big|\\ &= \sum_{k=1}^{n}|c_k|\mu(E_k) - \sum_{k=1}^{n}\Big|\int_{E_k}(g-h)\,d\mu\Big|\\ &= \int_E |g|\,d\mu - \sum_{k=1}^{n}\Big|\int_{E_k}(g-h)\,d\mu\Big|\\ &\geq \int_E |g|\,d\mu - \sum_{k=1}^{n}\int_{E_k}|g-h|\,d\mu\\ &\geq \int_E |h|\,d\mu - 2\varepsilon.\end{aligned}$$

The inequality above implies that $|\nu|(E) \geq \int_E |h|\,d\mu - 2\varepsilon$. Because ε is an arbitrary positive number, this implies $|\nu|(E) \geq \int_E |h|\,d\mu$, completing the proof. ■

Now we justify the terminology *total variation measure*.

9.11 *total variation measure is a measure*

Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. Then the total variation function $|\nu|$ is a (positive) measure on $(X, \mathcal{S})$.

Proof The definition of $|\nu|$ and 9.3(a) imply that $|\nu|(\varnothing) = 0$.

To show that $|\nu|$ is countably additive, suppose $A_1, A_2, \dots$ are disjoint sets in $\mathcal{S}$. Fix $m \in \mathbf{Z}^+$. For each $k \in \{1, \dots, m\}$, suppose $E_{1,k}, \dots, E_{n_k,k}$ are disjoint sets in $\mathcal{S}$ such that

9.12 $$E_{1,k} \cup \dots \cup E_{n_k,k} \subset A_k.$$

Then $\{E_{j,k} : 1 \leq k \leq m \text{ and } 1 \leq j \leq n_k\}$ is a disjoint collection of sets in $\mathcal{S}$ that are all contained in $\bigcup_{k=1}^{\infty} A_k$. Hence

$$\sum_{k=1}^{m}\sum_{j=1}^{n_k}|\nu(E_{j,k})| \leq |\nu|\Big(\bigcup_{k=1}^{\infty} A_k\Big).$$

Taking the supremum of the left side of the inequality above over all choices of $\{E_{j,k}\}$ satisfying 9.12 shows that

$$\sum_{k=1}^{m}|\nu|(A_k) \leq |\nu|\Big(\bigcup_{k=1}^{\infty} A_k\Big).$$

Because the inequality above holds for all $m \in \mathbf{Z}^+$, we have

$$\sum_{k=1}^{\infty}|\nu|(A_k) \leq |\nu|\Big(\bigcup_{k=1}^{\infty} A_k\Big).$$

To prove the inequality above in the other direction, suppose $E_1, \dots, E_n \in \mathcal{S}$ are disjoint sets such that $E_1 \cup \cdots \cup E_n \subset \bigcup_{k=1}^{\infty} A_k$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} |\nu|(A_k) &\geq \sum_{k=1}^{\infty} \sum_{j=1}^{n} |\nu(E_j \cap A_k)| \\ &= \sum_{j=1}^{n} \sum_{k=1}^{\infty} |\nu(E_j \cap A_k)| \\ &\geq \sum_{j=1}^{n} \Big| \sum_{k=1}^{\infty} \nu(E_j \cap A_k) \Big| \\ &= \sum_{j=1}^{n} |\nu(E_j)|,\end{aligned}$$

where the first line above follows from the definition of $|\nu|(A_k)$ and the last line above follows from the countable additivity of ν.

The inequality above and the definition of $|\nu|\big(\bigcup_{k=1}^{\infty} A_k\big)$ imply that

$$\sum_{k=1}^{\infty} |\nu|(A_k) \geq |\nu|\Big(\bigcup_{k=1}^{\infty} A_k\Big),$$

completing the proof. ■

The Banach Space of Measures

In this subsection, we make the set of complex or real measures on a measurable space into a vector space and then into a Banach space.

9.13 Definition ***addition and scalar multiplication of measures***

Suppose $(X, \mathcal{S})$ is a measurable space. For complex measures ν, μ on $(X, \mathcal{S})$ and $\alpha \in \mathbf{F}$, define complex measures $\nu + \mu$ and $\alpha\nu$ on $(X, \mathcal{S})$ by

$$(\nu + \mu)(E) = \nu(E) + \mu(E) \quad \text{and} \quad (\alpha\nu)(E) = \alpha\big(\nu(E)\big).$$

You should verify that if ν, μ, and α are as above, then $\nu + \mu$ and $\alpha\nu$ are complex measures on $(X, \mathcal{S})$. You should also verify that these natural definitions of addition and scalar multiplication make the set of complex (or real) measures on a measurable space $(X, \mathcal{S})$ into a vector space. We now introduce notation for this vector space.

9.14 Definition $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$

Suppose $(X, \mathcal{S})$ is a measurable space. Then $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ denotes the vector space of real measures on $(X, \mathcal{S})$ if $\mathbf{F} = \mathbf{R}$ and denotes the vector space of complex measures on $(X, \mathcal{S})$ if $\mathbf{F} = \mathbf{C}$.

We use the terminology *total variation norm* below even though we have not yet shown that the object being defined is a norm (especially because it is not obvious that $\|\nu\| < \infty$ for every complex measure ν). Soon we will justify this terminology.

9.15 Definition ***total variation norm of a complex measure***

Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. The total variation norm of ν, denoted $\|\nu\|$, is defined by

$$\|\nu\| = |\nu|(X).$$

9.16 Example *total variation norm*

- If μ is a finite (positive) measure, then $\|\mu\| = \mu(X)$, as you should verify.
- If μ is a (positive) measure, $h \in \mathcal{L}^1(\mu)$, and $d\nu = h\,d\mu$, then $\|\nu\| = \|h\|_1$ (as follows from 9.10).

The next result implies that if ν is a complex measure on a measurable space $(X, \mathcal{S})$, then $|\nu|(E) < \infty$ for every $E \in \mathcal{S}$.

9.17 ***total variation norm is finite***

Suppose $(X, \mathcal{S})$ is a measurable space and $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$. Then $\|\nu\| < \infty$.

Proof First consider the case where $\mathbf{F} = \mathbf{R}$. Thus ν is a real measure on $(X, \mathcal{S})$. To begin this proof by contradiction, suppose $\|\nu\| = |\nu|(X) = \infty$.

We inductively choose a decreasing sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$ as follows: Start by choosing $E_0 = X$. Now suppose $n \geq 0$ and $E_n \in \mathcal{S}$ has been chosen with $|\nu|(E_n) = \infty$ and $|\nu(E_n)| \geq n$. Because $|\nu|(E_n) = \infty$, 9.9 implies that there exists $A \in \mathcal{S}$ such that $A \subset E_n$ and $|\nu(A)| \geq n + 1 + |\nu(E_n)|$, which implies that

$$|\nu(E_n \setminus A)| = |\nu(E_n) - \nu(A)| \geq |\nu(A)| - |\nu(E_n)| \geq n + 1.$$

Now

$$|\nu|(A) + |\nu|(E_n \setminus A) = |\nu|(E_n) = \infty$$

because the total variation measure $|\nu|$ is a (positive) measure (by 9.11). The equation above shows that at least one of $|\nu|(A)$ and $|\nu|(E_n \setminus A)$ is ∞. Let $E_{n+1} = A$ if $|\nu|(A) = \infty$ and let $E_{n+1} = E_n \setminus A$ if $|\nu|(A) < \infty$. Thus $E_n \supset E_{n+1}$, $|\nu|(E_{n+1}) = \infty$, and $|\nu(E_{n+1})| \geq n + 1$.

Now 9.7(d) implies that $\nu\bigl(\bigcap_{n=1}^{\infty} E_n\bigr) = \lim_{n\to\infty} \nu(E_n)$. However, $|\nu(E_n)| \geq n$ for each $n \in \mathbf{Z}^+$, and thus the limit in the last equation does not exist (in $\mathbf{R}$). This contradiction completes the proof in the case where ν is a real measure.

Consider now the case where $\mathbf{F} = \mathbf{C}$; thus ν is a complex measure on $(X, \mathcal{S})$. Then

$$|\nu|(X) \leq |\operatorname{Re} \nu|(X) + |\operatorname{Im} \nu|(X) < \infty,$$

where the last inequality follows from applying the real case to $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$. ■

The previous result tells us that if $(X, \mathcal{S})$ is a measurable space, then $\|\nu\| < \infty$ for all $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$. This implies (as the reader should verify) that the total variation norm $\|\cdot\|$ is a norm on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$. The next result shows that this norm makes $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ into a Banach space (in other words, every Cauchy sequence in this norm converges).

9.18 ***the set of real or complex measures on $(X, \mathcal{S})$ is a Banach space***

Suppose $(X, \mathcal{S})$ is a measurable space. Then $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is a Banach space with the total variation norm.

Proof Suppose $\nu_1, \nu_2, \dots$ is a Cauchy sequence in $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$. For each $E \in \mathcal{S}$, we have

$$\begin{aligned} |\nu_j(E) - \nu_k(E)| &= |(\nu_j - \nu_k)(E)| \\ &\leq |\nu_j - \nu_k|(E) \\ &\leq \|\nu_j - \nu_k\|. \end{aligned}$$

Thus $\nu_1(E), \nu_2(E), \dots$ is a Cauchy sequence in $\mathbf{F}$ and hence converges. Thus we can define a function $\nu \colon \mathcal{S} \to \mathbf{F}$ by

$$\nu(E) = \lim_{j \to \infty} \nu_j(E).$$

To show that $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$, we must verify that ν is countably additive. To do this, suppose $E_1, E_2, \dots$ is a disjoint sequence of sets in $\mathcal{S}$. Let $\varepsilon > 0$. Let $m \in \mathbf{Z}^+$ be such that

9.19 $$\|\nu_j - \nu_k\| \leq \varepsilon \quad \text{for all } j, k \geq m.$$

If $n \in \mathbf{Z}^+$ is such that

9.20 $$\sum_{k=n}^{\infty} |\nu_m(E_k)| \leq \varepsilon$$

[such an n exists by applying 9.3(b) to ν_m] and if $j \geq m$, then

$$\begin{aligned} \sum_{k=n}^{\infty} |\nu_j(E_k)| &\leq \sum_{k=n}^{\infty} |(\nu_j - \nu_m)(E_k)| + \sum_{k=n}^{\infty} |\nu_m(E_k)| \\ &\leq \sum_{k=n}^{\infty} |\nu_j - \nu_m|(E_k) + \varepsilon \\ &= |\nu_j - \nu_m|\Big(\bigcup_{k=n}^{\infty} E_k\Big) + \varepsilon \\ &\leq 2\varepsilon, \end{aligned}$$ 9.21

where the second line uses 9.20, the third line uses the countable additivity of the measure $|\nu_j - \nu_m|$ (see 9.11), and the fourth line uses 9.19.

If ε and n are as in the paragraph above, then

$$\begin{aligned}\Big|\nu\Big(\bigcup_{k=1}^{\infty} E_k\Big) - \sum_{k=1}^{n-1} \nu(E_k)\Big| &= \Big|\lim_{j\to\infty} \nu_j\Big(\bigcup_{k=1}^{\infty} E_k\Big) - \lim_{j\to\infty} \sum_{k=1}^{n-1} \nu_j(E_k)\Big| \\ &= \lim_{j\to\infty}\Big|\sum_{k=n}^{\infty} \nu_j(E_k)\Big| \\ &\le 2\varepsilon,\end{aligned}$$

where the second line uses the countable additivity of the measure ν_j and the third line uses 9.21. The inequality above implies that $\nu\big(\bigcup_{k=1}^{\infty} E_k\big) = \sum_{k=1}^{\infty} \nu(E_k)$, completing the proof that $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

We still need to prove that $\lim_{k\to\infty}\|\nu - \nu_k\| = 0$. To do this, suppose $\varepsilon > 0$. Let $m \in \mathbf{Z}^+$ be such that

9.22 $$\|\nu_j - \nu_k\| \le \varepsilon \quad \text{for all } j, k \ge m.$$

Suppose $k \ge m$. Suppose also that $E_1, \dots, E_n \in \mathcal{S}$ are disjoint subsets of X. Then

$$\sum_{\ell=1}^{n} |(\nu - \nu_k)(E_\ell)| = \lim_{j\to\infty} \sum_{\ell=1}^{n} |(\nu_j - \nu_k)(E_\ell)| \le \varepsilon,$$

where the last inequality follows from 9.22 and the definition of the total variation norm. The inequality above implies that $\|\nu - \nu_k\| \le \varepsilon$, completing the proof. ■

EXERCISES 9A

1 Prove or give a counterexample: If ν is a real measure on a measurable space $(X, \mathcal{S})$ and $A, B \in \mathcal{S}$ are such that $\nu(A) \ge 0$ and $\nu(B) \ge 0$, then $\nu(A \cup B) \ge 0$.

2 Suppose ν is a real measure on $(X, \mathcal{S})$. Define $\mu \colon \mathcal{S} \to [0, \infty)$ by

$$\mu(E) = |\nu(E)|.$$

Prove that μ is a (positive) measure on $(X, \mathcal{S})$ if and only if the range of ν is contained in $[0, \infty)$ or the range of ν is contained in $(-\infty, 0]$.

3 Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. Prove that $|\nu|(X) = \nu(X)$ if and only if ν is a (positive) measure.

4 Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. Prove that if $E \in \mathcal{S}$ then

$$|\nu|(E) = \sup\Big\{\sum_{k=1}^{\infty} |\nu(E_k)| : E_1, E_2, \dots \text{ is a disjoint sequence in } \mathcal{S} \text{ such that } E = \bigcup_{k=1}^{\infty} E_k\Big\}.$$

5 Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and h is a nonnegative function in $\mathcal{L}^1(\mu)$. Let ν be the (positive) measure on $(X, \mathcal{S})$ defined by $d\nu = h\,d\mu$. Prove that

$$\int f\,d\nu = \int fh\,d\mu$$

for all $\mathcal{S}$-measurable functions $f\colon X \to [0, \infty]$.

6 Suppose $(X, \mathcal{S}, \mu)$ is a (positive) measure space. Prove that

$$\{h\,d\mu : h \in \mathcal{L}^1(\mu)\}$$

is a closed subspace of $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

7 (a) Suppose $\mathcal{B}$ is the collection of Borel subsets of $\mathbf{R}$. Show that the Banach space $\mathcal{M}_{\mathbf{F}}(\mathcal{B})$ is not separable.

(b) Give an example of a measurable space $(X, \mathcal{S})$ such that the Banach space $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is infinite-dimensional and separable.

8 Suppose $t > 0$ and λ is Lebesgue measure on the σ-algebra of Borel subsets of $[0, t]$. Suppose $h\colon [0, t] \to \mathbf{C}$ is the function defined by

$$h(x) = \cos x + i \sin x.$$

Let ν be the complex measure defined by $d\nu = h\,d\lambda$.

(a) Show that $\|\nu\| = t$.

(b) Show that if $E_1, E_2, \ldots$ is a sequence of disjoint Borel subsets of $[0, t]$, then

$$\sum_{k=1}^{\infty} |\nu(E_k)| < t.$$

[*This exercise shows that the supremum in the definition of* $|\nu|([0, t])$ *is not attained, even if countably many disjoint sets are allowed.*]

9 Give an example to show that 9.9 can fail if the hypothesis that ν is a real measure is replaced by the hypothesis that ν is a complex measure.

10 Suppose $(X, \mathcal{S})$ is a measurable space with $\mathcal{S} \neq \{\varnothing, X\}$. Prove that the total variation norm on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ does not come from an inner product. In other words, show that there does not exist an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ such that $\|\nu\| = \langle \nu, \nu \rangle^{1/2}$ for all $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$, where $\|\cdot\|$ is the usual total variation norm on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

11 For $(X, \mathcal{S})$ a measurable space and $b \in X$, define a finite (positive) measure δ_b on $(X, \mathcal{S})$ by

$$\delta_b(E) = \begin{cases} 1 & \text{if } b \in E, \\ 0 & \text{if } b \notin E \end{cases}$$

for $E \in \mathcal{S}$.

(a) Show that if $b, c \in X$, then $\|\delta_b + \delta_c\| = 2$.

(b) Give an example of a measurable space $(X, \mathcal{S})$ and $b, c \in X$ with $b \neq c$ such that $\|\delta_b - \delta_c\| \neq 2$.

9B Decomposition Theorems

Hahn Decomposition Theorem

The next result shows that a real measure on a measurable space $(X, \mathcal{S})$ decomposes X into two disjoint measurable sets such that every measurable subset of one of these two sets has nonnegative measure and every measurable subset of the other set has nonpositive measure.

The decomposition in the result below is not unique because a subset D of X with $|\nu|(D) = 0$ could be shifted from A to B or from B to A. However, Exercise 1 at the end of this section shows that the Hahn decomposition is almost unique.

9.23 *Hahn Decomposition Theorem*

Suppose ν is a real measure on a measurable space $(X, \mathcal{S})$. Then there exist sets $A, B \in \mathcal{S}$ such that

(a) $A \cup B = X$ and $A \cap B = \varnothing$;

(b) $\nu(E) \geq 0$ for every $E \in \mathcal{S}$ with $E \subset A$;

(c) $\nu(E) \leq 0$ for every $E \in \mathcal{S}$ with $E \subset B$.

9.24 **Example** *Hahn decomposition*

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$, $h \in \mathcal{L}^1(\mu)$ is real valued, and $d\nu = h\,d\mu$. Then a Hahn decomposition of the real measure ν is obtained by setting

$$A = \{x \in X : h(x) \geq 0\} \quad \text{and} \quad B = \{x \in X : h(x) < 0\}.$$

Proof of 9.23 Let

$$a = \sup\{\nu(E) : E \in \mathcal{S}\}.$$

Thus $a \leq \|\nu\| < \infty$, where the last inequality comes from 9.17. For each $j \in \mathbf{Z}^+$, let $A_j \in \mathcal{S}$ be such that

9.25 $$\nu(A_j) \geq a - \frac{1}{2^j}.$$

Temporarily fix $k \in \mathbf{Z}^+$. We will show by induction on n that if $n \in \mathbf{Z}^+$ with $n \geq k$, then

9.26 $$\nu\Bigl(\bigcup_{j=k}^{n} A_j\Bigr) \geq a - \sum_{j=k}^{n} \frac{1}{2^j}.$$

To get started with the induction, note that if $n = k$ then 9.26 holds because in this case 9.26 becomes 9.25. Now for the induction step, assume that $n \geq k$ and that 9.26 holds. Then

$$\begin{aligned}\nu\Big(\bigcup_{j=k}^{n+1} A_j\Big) &= \nu\Big(\bigcup_{j=k}^{n} A_j\Big) + \nu(A_{n+1}) - \nu\Big(\big(\bigcup_{j=k}^{n} A_j\big) \cap A_{n+1}\Big)\\ &\geq \Big(a - \sum_{j=k}^{n} \frac{1}{2^j}\Big) + \Big(a - \frac{1}{2^{n+1}}\Big) - a\\ &= a - \sum_{j=k}^{n+1} \frac{1}{2^j},\end{aligned}$$

where the first line follows from 9.7(b) and the second line follows from 9.25 and 9.26. We have now verified that 9.26 holds if n is replaced by $n+1$, completing the proof by induction of 9.26.

The sequence of sets $A_k, A_k \cup A_{k+1}, A_k \cup A_{k+1} \cup A_{k+2}, \dots$ is increasing. Thus taking the limit as $n \to \infty$ of both sides of 9.26 and using 9.7(c) gives

9.27 $$\nu\Big(\bigcup_{j=k}^{\infty} A_j\Big) \geq a - \frac{1}{2^{k-1}}.$$

Now let

$$A = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j.$$

The sequence of sets $\bigcup_{j=1}^{\infty} A_j, \bigcup_{j=2}^{\infty} A_j, \dots$ is decreasing. Thus 9.27 and 9.7(d) imply that $\nu(A) \geq a$. The definition of a now implies that

$$\nu(A) = a.$$

Suppose $E \in \mathcal{S}$ and $E \subset A$. Then $\nu(A) = a \geq \nu(A \setminus E)$. Thus we have $\nu(E) = \nu(A) - \nu(A \setminus E) \geq 0$, which proves (b).

Let $B = X \setminus A$; thus (a) holds. Suppose $E \in \mathcal{S}$ and $E \subset B$. Then we have $\nu(A \cup E) \leq a = \nu(A)$. Thus $\nu(E) = \nu(A \cup E) - \nu(A) \leq 0$, which proves (c). ∎

Jordan Decomposition Theorem

You should think of two complex or positive measures on a measurable space $(X, \mathcal{S})$ as being singular with respect to each other if the two measures live on different sets. Here is the formal definition.

9.28 Definition ***singular measures***

Suppose ν and μ are complex or positive measures on a measurable space $(X, \mathcal{S})$. Then ν and μ are called *singular* with respect to each other, denoted $\nu \perp \mu$, if there exist sets $A, B \in \mathcal{S}$ such that

- $A \cup B = X$ and $A \cap B = \varnothing$;
- $\nu(E) = \nu(E \cap A)$ and $\mu(E) = \mu(E \cap B)$ for all $E \in \mathcal{S}$.

9.29 **Example** *singular measures*

Suppose λ is Lebesgue measure on the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbf{R}$.

- Define positive measures ν, μ on $(\mathbf{R}, \mathcal{B})$ by
$$\nu(E) = |E \cap (-\infty, 0)| \quad \text{and} \quad \mu(E) = |E \cap (2, 3)|$$
for $E \in \mathcal{B}$. Then $\nu \perp \mu$ because ν lives on $(-\infty, 0)$ and μ lives on $[0, \infty)$. Neither ν nor μ is singular with respect to λ.

- Let $r_1, r_2, \ldots$ be a list of the rational numbers. Suppose $w_1, w_2, \ldots$ is a bounded sequence of complex numbers. Define a complex measure ν on $(\mathbf{R}, \mathcal{B})$ by
$$\nu(E) = \sum_{\{k \in \mathbf{Z}^+ : r_k \in E\}} \frac{w_k}{2^k}$$
for $E \in \mathcal{B}$. Then $\nu \perp \lambda$ because ν lives on $\mathbf{Q}$ and λ lives on $\mathbf{R} \setminus \mathbf{Q}$.

The hard work for proving the next result has already been done in proving the Hahn Decomposition Theorem (9.23).

9.30 ***Jordan Decomposition Theorem***

- Every real measure is the difference of two finite (positive) measures that are singular with respect to each other.

- More precisely, suppose ν is a real measure on a measurable space $(X, \mathcal{S})$. Then there exist unique finite (positive) measures ν^+ and ν^- on $(X, \mathcal{S})$ such that

9.31 $$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-.$$

Furthermore,
$$|\nu| = \nu^+ + \nu^-.$$

Proof Let $X = A \cup B$ be a Hahn decomposition of ν as in 9.23. Define functions $\nu^+ : \mathcal{S} \to [0, \infty)$ and $\nu^- : \mathcal{S} \to [0, \infty)$ by
$$\nu^+(E) = \nu(E \cap A) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap B).$$
The countable additivity of ν implies ν^+ and ν^- are finite (positive) measures on $(X, \mathcal{S})$, with $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

The definition of the total variation measure and 9.31 imply that $|\nu| = \nu^+ + \nu^-$, as you should verify.

Camille Jordan (1838–1922) is also known for certain matrices that are 0 except along the diagonal and the line above it.

The equations $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$ imply that
$$\nu^+ = \frac{|\nu| + \nu}{2} \quad \text{and} \quad \nu^- = \frac{|\nu| - \nu}{2}.$$
Thus the finite (positive) measures ν^+ and ν^- are uniquely determined by ν and the conditions in 9.31. ∎

Lebesgue Decomposition Theorem

The next definition captures the notion of one measure having more sets of measure 0 than another measure.

9.32 Definition ***absolutely continuous;*** $\ll$

Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$ and μ is a (positive) measure on $(X, \mathcal{S})$. Then ν is called *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if

$$\nu(E) = 0 \text{ for every set } E \in \mathcal{S} \text{ with } \mu(E) = 0.$$

9.33 Example *absolute continuity*

The reader should verify all the following examples:

- If μ is a (positive) measure and $h \in \mathcal{L}^1(\mu)$, then $h\,d\mu \ll \mu$.
- If ν is a real measure, then $\nu^+ \ll |\nu|$ and $\nu^- \ll |\nu|$.
- If ν is a complex measure, then $\nu \ll |\nu|$.
- If ν is a complex measure, then $\operatorname{Re}\nu \ll |\nu|$ and $\operatorname{Im}\nu \ll |\nu|$.
- Every measure on a measurable space $(X, \mathcal{S})$ is absolutely continuous with respect to counting measure on $(X, \mathcal{S})$.

The next result should help you think that absolute continuity and singularity are two extreme possibilities for the relationship between two complex measures.

9.34 ***absolutely continuous and singular implies*** 0 ***measure***

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$. Then the only complex measure on $(X, \mathcal{S})$ that is both absolutely continuous and singular with respect to μ is the 0 measure.

Proof Suppose ν is a complex measure on $(X, \mathcal{S})$ such that $\nu \ll \mu$ and $\nu \perp \mu$. Thus there exist sets $A, B \in \mathcal{S}$ such that $A \cup B = X$, $A \cap B = \varnothing$, and $\nu(E) = \nu(E \cap A)$ and $\mu(E) = \mu(E \cap B)$ for every $E \in \mathcal{S}$.

Suppose $E \in \mathcal{S}$. Then

$$\mu(E \cap A) = \mu\big((E \cap A) \cap B\big) = \mu(\varnothing) = 0.$$

Because $\nu \ll \mu$, this implies that $\nu(E \cap A) = 0$. Thus $\nu(E) = 0$. Hence ν is the 0 measure. ■

Our next result states that a (positive) measure on a measurable space $(X, \mathcal{S})$ determines a decomposition of each complex measure on $(X, \mathcal{S})$ as the sum of the two extreme types of complex measures (absolute continuity and singularity).

9.35 *Lebesgue Decomposition Theorem*

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$.

- Every complex measure on $(X, \mathcal{S})$ is the sum of a complex measure absolutely continuous with respect to μ and a complex measure singular with respect to μ.

- More precisely, suppose ν is a complex measure on $(X, \mathcal{S})$. Then there exist unique complex measures ν_a and ν_s on $(X, \mathcal{S})$ such that $\nu = \nu_a + \nu_s$ and

$$\nu_a \ll \mu \quad \text{and} \quad \nu_s \perp \mu.$$

Proof Let

$$b = \sup\{|\nu|(B) : B \in \mathcal{S} \text{ and } \mu(B) = 0\}.$$

For each $k \in \mathbf{Z}^+$, let $B_k \in \mathcal{S}$ be such that

$$|\nu|(B_k) \geq b - \tfrac{1}{k} \quad \text{and} \quad \mu(B_k) = 0.$$

Let

$$B = \bigcup_{k=1}^{\infty} B_k.$$

Then $\mu(B) = 0$ and $|\nu|(B) = b$.

Let $A = X \setminus B$. Define complex measures ν_a and ν_s on $(X, \mathcal{S})$ by

$$\nu_a(E) = \nu(E \cap A) \quad \text{and} \quad \nu_s(E) = \nu(E \cap B).$$

Clearly $\nu = \nu_a + \nu_s$.

If $E \in \mathcal{S}$, then

$$\mu(E) = \mu(E \cap A) + \mu(E \cap B) = \mu(E \cap A),$$

where the last equality holds because $\mu(B) = 0$. The equation above implies that $\nu_s \perp \mu$.

To prove that $\nu_a \ll \mu$, suppose $E \in \mathcal{S}$ and $\mu(E) = 0$. Then $\mu(B \cup E) = 0$ and hence

$$b \geq |\nu|(B \cup E) = |\nu|(B) + |\nu|(E \setminus B) = b + |\nu|(E \setminus B),$$

which implies that $|\nu|(E \setminus B) = 0$. Thus

$$\nu_a(E) = \nu(E \cap A) = \nu(E \setminus B) = 0,$$

which implies that $\nu_a \ll \mu$.

The construction of ν_a and ν_s shows that if ν is a positive (or real) measure, then so are ν_a and ν_s.

We have now proved all parts of this result except the uniqueness of the Lebesgue decomposition. To prove the uniqueness, suppose ν_1 and ν_2 are complex measures on $(X, \mathcal{S})$ such that $\nu_1 \ll \mu$, $\nu_2 \perp \mu$, and $\nu = \nu_1 + \nu_2$. Then

$$\nu_1 - \nu_a = \nu_s - \nu_2.$$

The left side of the equation above is absolutely continuous with respect to μ and the right side is singular with respect to μ. Thus both sides are both absolutely continuous and singular with respect to μ. Thus 9.34 implies that $\nu_1 = \nu_a$ and $\nu_2 = \nu_s$. ■

Radon–Nikodym Theorem

If μ is a (positive) measure, $h \in \mathcal{L}^1(\mu)$, and $d\nu = h\,d\mu$, then $\nu \ll \mu$. The next result gives the important converse—if μ is σ-finite, then every complex measure that is absolutely continuous with respect to μ is of the form $h\,d\mu$ for some $h \in \mathcal{L}^1(\mu)$. The hypothesis that μ is σ-finite cannot be deleted.

The result below was first proved by Radon and Otto Nikodym (1887–1974).

9.36 *Radon–Nikodym Theorem*

Suppose μ is a (positive) σ-finite measure on a measurable space $(X, \mathcal{S})$. Suppose ν is a complex measure on $(X, \mathcal{S})$ such that $\nu \ll \mu$. Then there exists $h \in \mathcal{L}^1(\mu)$ such that $d\nu = h\,d\mu$.

Proof First consider the case where both μ and ν are finite (positive) measures. Define $\varphi\colon L^2(\nu+\mu) \to \mathbf{R}$ by

9.37 $$\varphi(f) = \int f\,d\nu.$$

To show that φ is well defined, first note that if $f \in \mathcal{L}^2(\nu+\mu)$, then

9.38 $$\int |f|\,d\nu \le \int |f|\,d(\nu+\mu) \le \big(\nu(X)+\mu(X)\big)^{1/2}\|f\|_{L^2(\nu+\mu)} < \infty,$$

where the middle inequality follows from Hölder's inequality (7.9) applied to the functions 1 and f. Now 9.38 shows that $\int f\,d\nu$ makes sense for $f \in \mathcal{L}^2(\nu+\mu)$. Furthermore, if two functions in $\mathcal{L}^2(\nu+\mu)$ differ only on a set of $(\nu+\mu)$-measure 0, then they differ only on a set of ν-measure 0. Thus φ as defined in 9.37 makes sense as a linear functional on $L^2(\nu+\mu)$.

Because $|\varphi(f)| \le \int |f|\,d\nu$, 9.38 shows that φ is a bounded linear functional on $L^2(\nu+\mu)$. The Riesz Representation Theorem (8.47) now implies that there exists $g \in \mathcal{L}^2(\nu+\mu)$ such that

The clever idea of using Hilbert space techniques in this proof comes from John von Neumann (1903–1957).

$$\int f\,d\nu = \int fg\,d(\nu+\mu)$$

for all $f \in \mathcal{L}^2(\nu+\mu)$. Hence

9.39 $$\int f(1-g)\,d\nu = \int fg\,d\mu$$

for all $f \in \mathcal{L}^2(\nu+\mu)$.

If f equals the characteristic function of $\{x \in X : g(x) \ge 1\}$, then the left side of 9.39 is less than or equal to 0 and the right side of 9.39 is greater than or equal to 0; hence both sides are 0. Thus $\int fg\,d\mu = 0$, which implies (with this choice of f) that $\mu(\{x \in X : g(x) \ge 1\}) = 0$.

Similarly, if f equals the characteristic function of $\{x \in X : g(x) < 0\}$, then the left side of 9.39 is greater than or equal to 0 and the right side of 9.39 is less than or equal to 0; hence both sides are 0. Thus $\int fg\,d\mu = 0$, which implies (with this choice of f) that $\mu(\{x \in X : g(x) < 0\}) = 0$.

Because $\nu \ll \mu$, the two previous paragraphs imply that

$$\nu(\{x \in X : g(x) \geq 1\}) = 0 \quad \text{and} \quad \nu(\{x \in X : g(x) < 0\}) = 0.$$

Thus we can modify g (for example by redefining g to be $\frac{1}{2}$ on the two sets appearing above; both those sets have ν-measure 0 and μ-measure 0) and from now on we can assume that $0 \leq g(x) < 1$ for all $x \in X$ and that 9.39 holds for all $f \in \mathcal{L}^2(\nu + \mu)$. Hence we can define $h\colon X \to [0, \infty)$ by

$$h(x) = \frac{g(x)}{1 - g(x)}.$$

Suppose $E \in \mathcal{S}$. For each $k \in \mathbf{Z}^+$, let

$$f_k(x) = \begin{cases} \frac{\chi_E(x)}{1-g(x)} & \text{if } \frac{\chi_E(x)}{1-g(x)} \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Taking $f = \chi_E/(1-g)$ in 9.39 would give $\nu(E) = \int_E h\,d\mu$, but this function f might not be in $\mathcal{L}^2(\nu + \mu)$ and thus we need to be a bit more careful.

Then $f_k \in \mathcal{L}^2(\nu + \mu)$. Now 9.39 implies

$$\int f_k(1 - g)\,d\nu = \int f_k g\,d\mu.$$

Taking the limit as $k \to \infty$ and using the Monotone Convergence Theorem (3.11) shows that

9.40 $$\int_E 1\,d\nu = \int_E h\,d\mu.$$

Thus $d\nu = h\,d\mu$, completing the proof in the case where both ν and μ are (positive) finite measures [note that $h \in \mathcal{L}^1(\mu)$ because h is a nonnegative function and we can take $E = X$ in the equation above].

Now relax the assumption on μ to the hypothesis that μ is a σ-finite measure. Thus there exists an increasing sequence $X_1 \subset X_2 \subset \cdots$ of sets in $\mathcal{S}$ such that $\bigcup_{k=1}^{\infty} X_k = X$ and $\mu(X_k) < \infty$ for each $k \in \mathbf{Z}^+$. For $k \in \mathbf{Z}^+$, let ν_k and μ_k denote the restrictions of ν and μ to the σ-algebra on X_k consisting of those sets in $\mathcal{S}$ that are subsets of X_k. Then $\nu_k \ll \mu_k$. Thus by the case we have already proved, there exists a nonnegative function $h_k \in \mathcal{L}^1(\mu_k)$ such that $d\nu_k = h_k\,d\mu_k$. If $j < k$, then

$$\int_E h_j\,d\mu = \nu(E) = \int_E h_k\,d\mu$$

for every set $E \in \mathcal{S}$ with $E \subset X_j$; thus $\mu(\{x \in X_j : h_j(x) \neq h_k(x)\}) = 0$. Hence there exists an $\mathcal{S}$-measurable function $h\colon X \to [0, \infty)$ such that

$$\mu(\{x \in X_k : h(x) \neq h_k(x)\}) = 0$$

for every $k \in \mathbf{Z}^+$. The Monotone Convergence Theorem (3.11) can now be used to show that 9.40 holds for every $E \in \mathcal{S}$. Thus $d\nu = h\,d\mu$, completing the proof in the case where ν is a (positive) finite measure.

Now relax the assumption on ν to the assumption that ν is a real measure. The measure ν equals one-half the difference of the two positive (finite) measures $|\nu| + \nu$ and $|\nu| - \nu$, each of which is absolutely continuous with respect to μ. By the case proved in the previous paragraph, there exist $h_+, h_- \in \mathcal{L}^1(\mu)$ such that

$$d(|\nu| + \nu) = h_+ \, d\mu \quad \text{and} \quad d(|\nu| - \nu) = h_- \, d\mu.$$

Taking $h = \frac{1}{2}(h_+ - h_-)$, we have $d\nu = h \, d\mu$, completing the proof in the case where ν is a real measure.

Finally, if ν is a complex measure, apply the result in the previous paragraph to the real measures $\operatorname{Re} \nu$, $\operatorname{Im} \nu$, producing $h_{\mathrm{Re}}, h_{\mathrm{Im}} \in \mathcal{L}^1(\mu)$ such that $d(\operatorname{Re} \nu) = h_{\mathrm{Re}} \, d\mu$ and $d(\operatorname{Im} \nu) = h_{\mathrm{Im}} \, d\mu$. Taking $h = h_{\mathrm{Re}} + i h_{\mathrm{Im}}$, we have $d\nu = h \, d\mu$, completing the proof in the case where ν is a complex measure. ■

The function h provided by the Radon–Nikodym Theorem is unique up to changes on sets with μ-measure 0. If we think of h as an element of $L^1(\mu)$ instead of $\mathcal{L}^1(\mu)$, then the choice of h is unique.

When $d\nu = h \, d\mu$, the notation $h = \frac{d\nu}{d\mu}$ is used by some authors, and h is called the *Radon–Nikodym derivative* of ν with respect to μ.

The next result is a nice consequence of the Radon–Nikodym Theorem.

9.41 *if ν is a complex measure, then* $d\nu = h \, d|\nu|$ *for some* h *with* $|h(x)| = 1$

(a) Suppose ν is a real measure on a measurable space $(X, \mathcal{S})$. Then there exists an $\mathcal{S}$-measurable function $h \colon X \to \{-1, 1\}$ such that $d\nu = h \, d|\nu|$.

(b) Suppose ν is a complex measure on a measurable space $(X, \mathcal{S})$. Then there exists an $\mathcal{S}$-measurable function $h \colon X \to \{z \in \mathbf{C} : |z| = 1\}$ such that $d\nu = h \, d|\nu|$.

Proof Because $\nu \ll |\nu|$, the Radon–Nikodym Theorem (9.36) tells us that there exists $h \in \mathcal{L}^1(|\nu|)$ (with h real valued if ν is a real measure) such that $d\nu = h \, d|\nu|$. Now 9.10 implies that $d|\nu| = |h| \, d|\nu|$, which implies that $|h| = 1$ almost everywhere (with respect to $|\nu|$). Refine h to be 1 on the set $\{x \in X : |h(x)| \neq 1\}$, which gives the desired result. ■

We could have proved part (a) of the result above by taking $h = \chi_A - \chi_B$ in the Hahn Decomposition Theorem (9.23).

Conversely, we could give a new proof of the Hahn Decomposition Theorem by using part (a) of the result above and taking

$$A = \{x \in X : h(x) = 1\} \quad \text{and} \quad B = \{x \in X : h(x) = -1\}.$$

We could also give a new proof of the Jordan Decomposition Theorem (9.30) by using part (a) of the result above and taking

$$\nu^+ = \chi_{\{x \in X \,:\, h(x) = 1\}} \, d|\nu| \quad \text{and} \quad \nu^- = \chi_{\{x \in X \,:\, h(x) = -1\}} \, d|\nu|.$$

Dual Space of $L^p(\mu)$

Recall that the dual space of a normed vector space V is the Banach space of bounded linear functionals on V; the dual space of V is denoted by V'. Recall also that if $1 \leq p \leq \infty$, then the dual exponent p' is defined by the equation $\frac{1}{p} + \frac{1}{p'} = 1$.

The dual space of ℓ^p can be identified with $\ell^{p'}$ for $1 \leq p < \infty$, as we saw in 7.26. We are now ready to prove the analogous result for an arbitrary (positive) measure, identifying the dual space of $L^p(\mu)$ with $L^{p'}(\mu)$ [with the mild restriction that μ is σ-finite if $p = 1$]. In the special case where μ is counting measure on $\mathbf{Z}^+$, this new result reduces to the previous result about ℓ^p.

For $1 < p < \infty$, the next result differs from 7.25 by only one word, with "to" in 7.25 changed to "onto" below. Thus we already know (and will use in the proof) that the map $h \mapsto \varphi_h$ is a one-to-one linear map from $L^{p'}(\mu)$ to $\bigl(L^p(\mu)\bigr)'$ and that $\|\varphi_h\| = \|h\|_{p'}$ for all $h \in L^{p'}(\mu)$. The new aspect of the result below is the assertion that every bounded linear functional on $L^p(\mu)$ is of the form φ_h for some $h \in L^{p'}(\mu)$. The key tool we use in proving this new assertion is the Radon–Nikodym Theorem.

9.42 *dual space of $L^p(\mu)$ is $L^{p'}(\mu)$*

Suppose μ is a (positive) measure and $1 \leq p < \infty$ [with the additional hypothesis that μ is a σ-finite measure if $p = 1$]. For $h \in L^{p'}(\mu)$, define $\varphi_h \colon L^p(\mu) \to \mathbf{F}$ by

$$\varphi_h(f) = \int fh\,d\mu.$$

Then $h \mapsto \varphi_h$ is a one-to-one linear map from $L^{p'}(\mu)$ onto $\bigl(L^p(\mu)\bigr)'$. Furthermore, $\|\varphi_h\| = \|h\|_{p'}$ for all $h \in L^{p'}(\mu)$.

Proof The case $p = 1$ is left to the reader as an exercise. Thus assume that $1 < p < \infty$.

Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and φ is a bounded linear functional on $L^p(\mu)$; in other words, suppose $\varphi \in \bigl(L^p(\mu)\bigr)'$.

Consider first the case where μ is a finite (positive) measure. Define a function $\nu \colon \mathcal{S} \to \mathbf{F}$ by

$$\nu(E) = \varphi(\chi_E).$$

If $E_1, E_2, \ldots$ are disjoint sets in $\mathcal{S}$, then

$$\nu\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \varphi\Bigl(\chi_{\bigcup_{k=1}^{\infty} E_k}\Bigr) = \varphi\Bigl(\sum_{k=1}^{\infty} \chi_{E_k}\Bigr) = \sum_{k=1}^{\infty} \varphi(\chi_{E_k}) = \sum_{k=1}^{\infty} \nu(E_k),$$

where the infinite sum in the third term converges in the $L^p(\mu)$-norm to $\chi_{\bigcup_{k=1}^{\infty} E_k}$, and the third equality holds because φ is a continuous linear functional. The equation above shows that ν is countably additive. Thus ν is a complex measure on $(X, \mathcal{S})$ [and is a real measure if $\mathbf{F} = \mathbf{R}$].

If $E \in \mathcal{S}$ and $\mu(E) = 0$, then χ_E is the 0 element of $L^p(\mu)$, which implies that $\varphi(\chi_E) = 0$, which means that $\nu(E) = 0$. Hence $\nu \ll \mu$. By the Radon–Nikodym Theorem (9.36), there exists $h \in \mathcal{L}^1(\mu)$ such that $d\nu = h\,d\mu$. Hence

$$\varphi(\chi_E) = \nu(E) = \int_E h\,d\mu = \int \chi_E h\,d\mu$$

for every $E \in \mathcal{S}$. The equation above, along with the linearity of φ, implies that

9.43 $$\varphi(f) = \int fh\,d\mu \quad \text{for every simple } \mathcal{S}\text{-measurable function } f\colon X \to \mathbf{F}.$$

Because every bounded $\mathcal{S}$-measurable function is the uniform limit on X of a sequence of simple $\mathcal{S}$-measurable functions (see 2.89), we can conclude from 9.43 that

9.44 $$\varphi(f) = \int fh\,d\mu \quad \text{for every } f \in L^\infty(\mu).$$

For $k \in \mathbf{Z}^+$, let

$$E_k = \{x \in X : 0 < |h(x)| \le k\}$$

and define $f_k \in L^p(\mu)$ by

9.45 $$f_k(x) = \begin{cases} \overline{h(x)}\,|h(x)|^{p'-2} & \text{if } x \in E_k, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\int |h|^{p'} \chi_{E_k}\,d\mu = \varphi(f_k) \le \|\varphi\|\,\|f_k\|_p = \|\varphi\| \Big(\int |h|^{p'} \chi_{E_k}\,d\mu\Big)^{1/p},$$

where the first equality follows from 9.44 and 9.45, and the last equality follows from 9.45 [which implies that $|f_k(x)|^p = |h(x)|^{p'} \chi_{E_k}(x)$ for $x \in X$]. After dividing by $\big(\int |h|^{p'} \chi_{E_k}\,d\mu\big)^{1/p}$, the inequality between the first and last terms above becomes

$$\|h\chi_{E_k}\|_{p'} \le \|\varphi\|.$$

Taking the limit as $k \to \infty$ shows, via the Monotone Convergence Theorem (3.11), that

$$\|h\|_{p'} \le \|\varphi\|.$$

Thus $h \in L^{p'}(\mu)$. Because each $f \in L^p(\mu)$ can be approximated in the $L^p(\mu)$ norm by functions in $L^\infty(\mu)$, 9.44 now shows that $\varphi = \varphi_h$, completing the proof in the case where μ is a finite (positive) measure.

Now relax the assumption that μ is a finite (positive) measure to the hypothesis that μ is a (positive) measure. For $E \in \mathcal{S}$, let $\mathcal{S}_E = \{A \in \mathcal{S} : A \subset E\}$ and let μ_E be the (positive) measure on $(E, \mathcal{S}_E)$ defined by $\mu_E(A) = \mu(A)$ for $A \in \mathcal{S}_E$. We can identify $L^p(\mu_E)$ with the subspace of functions in $L^p(\mu)$ that vanish (almost everywhere) outside E. With this identification, let $\varphi_E = \varphi|_{L^p(\mu_E)}$. Then φ_E is a bounded linear functional on $L^p(\mu_E)$ and $\|\varphi_E\| \le \|\varphi\|$.

If $E \in \mathcal{S}$ and $\mu(E) < \infty$, then the finite measure case that we have already proved as applied to φ_E implies that there exists a unique $h_E \in L^{p'}(\mu_E)$ such that

9.46 $$\varphi(f) = \int_E f h_E \, d\mu \quad \text{for all } f \in L^p(\mu_E).$$

If $D, E \in \mathcal{S}$ and $D \subset E$ with $\mu(E) < \infty$, then $h_D(x) = h_E(x)$ for almost every $x \in D$ (use the uniqueness part of the result).

For each $k \in \mathbf{Z}^+$, there exists $f_k \in L^p(\mu)$ such that

9.47 $$\|f_k\|_p \leq 1 \quad \text{and} \quad |\varphi(f_k)| > \|\varphi\| - \tfrac{1}{k}.$$

The Dominated Convergence Theorem (3.31) implies that

$$\lim_{n\to\infty} \big\|f_k \chi_{\{x \in X : |f_k(x)| > \frac{1}{n}\}} - f_k\big\|_p = 0$$

for each $k \in \mathbf{Z}^+$. Thus we can replace f_k by $f_k \chi_{\{x \in X : |f_k(x)| > \frac{1}{n}\}}$ for sufficiently large n and still have 9.47 hold. In other words, for each $k \in \mathbf{Z}^+$, we can assume that there exists $n_k \in \mathbf{Z}^+$ such that for each $x \in X$, either $|f_k(x)| > 1/n_k$ or $f_k(x) = 0$.

Set $D_k = \{x \in X : |f_k(x)| > 1/n_k\}$. Then $\mu(D_k) < \infty$ [because $f_k \in L^p(\mu)$] and

9.48 $$f_k(x) = 0 \text{ for all } x \in X \setminus D_k.$$

For $k \in \mathbf{Z}^+$, let $E_k = D_1 \cup \cdots \cup D_k$. Because $E_1 \subset E_2 \subset \cdots$, we see that if $j < k$, then $h_{E_j}(x) = h_{E_k}(x)$ for almost every $x \in E_j$. Also, 9.47 and 9.48 imply that

9.49 $$\lim_{k\to\infty} \|h_{E_k}\|_{p'} = \lim_{k\to\infty} \|\varphi_{E_k}\| = \|\varphi\|.$$

Let $E = \bigcup_{k=1}^{\infty} E_k$. Let h be the function that equals h_{E_k} almost everywhere on E_k for each $k \in \mathbf{Z}^+$ and equals 0 on $X \setminus E$. The Monotone Convergence Theorem and 9.49 show that

$$\|h\|_{p'} = \|\varphi\|.$$

If $f \in L^p(\mu_E)$, then $\lim_{k\to\infty} \|f - f\chi_{E_k}\|_p = 0$ by the Dominated Convergence Theorem. Thus if $f \in L^p(\mu_E)$, then

9.50 $$\varphi(f) = \lim_{k\to\infty} \varphi(f\chi_{E_k}) = \lim_{k\to\infty} \int f\chi_{E_k} h \, d\mu = \int f h \, d\mu,$$

where the first equality follows from the continuity of φ, the second equality follows from 9.46 as applied to each E_k [valid because $\mu(E_k) < \infty$], and the third equality follows from the Dominated Convergence Theorem.

If D is an $\mathcal{S}$-measurable subset of $X \setminus E$ with $\mu(D) < \infty$, then $\|h_D\|_{p'} = 0$ because otherwise we would have $\|h + h_D\|_{p'} > \|h\|_{p'}$ and the linear functional on $L^p(\mu)$ induced by $h + h_D$ would have norm larger than $\|\varphi\|$ even though it agrees with φ on $L^p(\mu_{E\cup D})$. Because $\|h_D\|_{p'} = 0$, we see from 9.50 that $\varphi(f) = \int f h \, d\mu$ for all $f \in L^p(\mu_{E\cup D})$.

Every element of $L^p(\mu)$ can be approximated in norm by elements of $L^p(\mu_E)$ plus functions that live on subsets of $X \setminus E$ with finite measure. Thus the previous paragraph implies that $\varphi(f) = \int f h \, d\mu$ for all $f \in L^p(\mu)$, completing the proof. ■

EXERCISES 9B

1 Suppose ν is a real measure on a measurable space $(X, \mathcal{S})$. Prove that the Hahn decomposition of ν is almost unique, in the sense that if A, B and A', B' are pairs satisfying the Hahn Decomposition Theorem (9.23), then

$$|\nu|(A \setminus A') = |\nu|(A' \setminus A) = |\nu|(B \setminus B') = |\nu|(B' \setminus B) = 0.$$

2 Suppose μ is a (positive) measure and $g, h \in \mathcal{L}^1(\mu)$. Prove that $g\,d\mu \perp h\,d\mu$ if and only if $g(x)h(x) = 0$ for almost every $x \in X$.

3 Suppose ν and μ are complex measures on a measurable space $(X, \mathcal{S})$. Show that the following are equivalent:

(a) $\nu \perp \mu$.

(b) $|\nu| \perp |\mu|$.

(c) $\operatorname{Re} \nu \perp \mu$ and $\operatorname{Im} \nu \perp \mu$.

4 Suppose ν and μ are complex measures on a measurable space $(X, \mathcal{S})$. Prove that if $\nu \perp \mu$, then $|\nu + \mu| = |\nu| + |\mu|$ and $\|\nu + \mu\| = \|\nu\| + \|\mu\|$.

5 Suppose ν and μ are finite (positive) measures. Prove that $\nu \perp \mu$ if and only if $\|\nu - \mu\| = \|\nu\| + \|\mu\|$.

6 Suppose μ is a complex or positive measure on a measurable space $(X, \mathcal{S})$. Prove that

$$\{\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S}) : \nu \perp \mu\}$$

is a closed subspace of $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

7 Use the Cantor set to prove that there exists a (positive) measure ν on $(\mathbf{R}, \mathcal{B})$ such that $\nu \perp \lambda$ and $\nu(\mathbf{R}) \neq 0$ but $\nu(\{x\}) = 0$ for every $x \in \mathbf{R}$; here λ denotes Lebesgue measure on the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbf{R}$.
[*The second bullet point in Example 9.29 does not provide an example of the desired behavior because in that example, $\nu(\{r_k\}) \neq 0$ for all $k \in \mathbf{Z}^+$ with $w_k \neq 0$.*]

8 Suppose ν is a real measure on a measurable space $(X, \mathcal{S})$. Prove that

$$\nu^+(E) = \sup\{\nu(D) : D \in \mathcal{S} \text{ and } D \subset E\}$$

and

$$\nu^-(E) = -\inf\{\nu(D) : D \in \mathcal{S} \text{ and } D \subset E\}.$$

9 Suppose μ is a (positive) finite measure on a measurable space $(X, \mathcal{S})$ and h is a nonnegative function in $\mathcal{L}^1(\mu)$. Thus $h\,d\mu \ll d\mu$. Find a reasonable condition on h that is equivalent to the condition $d\mu \ll h\,d\mu$.

10 Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and ν is a complex measure on $(X, \mathcal{S})$. Show that the following are equivalent:

(a) $\nu \ll \mu$.

(b) $|\nu| \ll \mu$.

(c) $\operatorname{Re} \nu \ll \mu$ and $\operatorname{Im} \nu \ll \mu$.

11 Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and ν is a real measure on $(X, \mathcal{S})$. Show that $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

12 Suppose μ is a (positive) measure on a measurable space $(X, \mathcal{S})$. Prove that

$$\{\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S}) : \nu \ll \mu\}$$

is a closed subspace of $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

13 Give an example to show that the Radon–Nikodym Theorem (9.36) can fail if the σ-finite hypothesis is eliminated.

14 Suppose μ is a (positive) σ-finite measure on a measurable space $(X, \mathcal{S})$ and ν is a complex measure on $(X, \mathcal{S})$. Show that the following are equivalent:

(a) $\nu \ll \mu$.

(b) for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ for every set $E \in \mathcal{S}$ with $\mu(E) < \delta$.

(c) for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu|(E) < \varepsilon$ for every set $E \in \mathcal{S}$ with $\mu(E) < \delta$.

15 Prove 9.42 [with the extra hypothesis that μ is a σ-finite (positive) measure] in the case where $p = 1$.

16 Explain where the proof of 9.42 fails if $p = \infty$.

17 Prove that if μ is a (positive) measure and $1 < p < \infty$, then $L^p(\mu)$ is reflexive. [*See the definition before Exercise 19 in Section 7B for the meaning of* reflexive.]

18 Prove that $L^1(\mathbf{R})$ is not reflexive.

Chapter 10
Linear Maps on Hilbert Spaces

A special tool called the *adjoint* helps provide insight into the behavior of linear maps on Hilbert spaces. This chapter begins with a study of the adjoint and its connection to the null space and range of a linear map.

Then we discuss various issues connected with the invertibility of operators on Hilbert spaces. These issues lead to the spectrum, which is a set of numbers that gives important information about an operator.

This chapter then looks at special classes of operators on Hilbert spaces: self-adjoint operators, normal operators, isometries, unitary operators, integral operators, and compact operators.

Even on infinite-dimensional Hilbert spaces, compact operators display many characteristics expected from finite-dimensional linear algebra. We will see that the powerful Spectral Theorem for compact operators greatly resembles the finite-dimensional version. Also, we develop the Singular Value Decomposition for an arbitrary compact operator, again quite similar to the finite-dimensional result.



The Botanical Garden at Uppsala University (the oldest university in Sweden, founded in 1477), where Erik Fredholm (1866–1927) was a student. The theorem called the Fredholm Alternative, which we prove in this chapter, states that a compact operator minus a nonzero scalar multiple of the identity operator is injective if and only if it is surjective.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_10

10A Adjoints and Invertibility

Adjoints of Linear Maps on Hilbert Spaces

The next definition provides a key tool for studying linear maps on Hilbert spaces.

10.1 Definition ***adjoint;*** T^*

Suppose V and W are Hilbert spaces and $T\colon V \to W$ is a bounded linear map. The *adjoint* of T is the function $T^*\colon W \to V$ such that

$$\langle Tf, g\rangle = \langle f, T^*g\rangle$$

for every $f \in V$ and every $g \in W$.

The word **adjoint** *has two unrelated meanings in linear algebra. We need only the meaning defined above.*

To see why the definition above makes sense, fix $g \in W$. Consider the linear functional on V defined by $f \mapsto \langle Tf, g\rangle$. This linear functional is bounded because

$$|\langle Tf, g\rangle| \le \|Tf\|\,\|g\| \le \|T\|\,\|g\|\,\|f\|$$

for all $f \in V$; thus the linear functional $f \mapsto \langle Tf, g\rangle$ has norm at most $\|T\|\,\|g\|$. By the Riesz Representation Theorem (8.47), there exists a unique element of V (with norm at most $\|T\|\,\|g\|$) such that this linear functional is given by taking the inner product with it. We call this unique element T^*g. In other words, T^*g is the unique element of V such that

10.2
$$\langle Tf, g\rangle = \langle f, T^*g\rangle$$

for every $f \in V$. Furthermore,

10.3
$$\|T^*g\| \le \|T\|\|g\|.$$

In 10.2, notice that the inner product on the left is the inner product in W and the inner product on the right is the inner product in V.

10.4 Example *multiplication operators*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h \in \mathcal{L}^\infty(\mu)$. Define the *multiplication operator* $M_h\colon L^2(\mu) \to L^2(\mu)$ by

$$M_h f = fh.$$

Then M_h is a bounded linear map and $\|M_h\| \le \|h\|_\infty$. Because

$$\langle M_h f, g\rangle = \int fh\overline{g}\,d\mu = \langle f, M_{\overline{h}}g\rangle$$

for all $f, g \in L^2(\mu)$, we have $M_h{}^* = M_{\overline{h}}$.

The complex conjugates that appear in this example are unnecessary (but they do no harm) if $\mathbf{F} = \mathbf{R}$.

10.5 Example *linear maps induced by integration*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces and $K \in \mathcal{L}^2(\mu \times \nu)$. Define a linear map $\mathcal{I}_K \colon L^2(\nu) \to L^2(\mu)$ by

10.6
$$(\mathcal{I}_K f)(x) = \int_Y K(x, y) f(y) \, d\nu(y)$$

for $f \in L^2(\nu)$ and $x \in X$. To see that this definition makes sense, first note that there are no worrisome measurability issues because for each $x \in X$, the function $y \mapsto K(x, y)$ is a $\mathcal{T}$-measurable function on Y (see 5.9).

Suppose $f \in L^2(\nu)$. Use the Cauchy–Schwarz inequality (8.11) or Hölder's inequality (7.9) to show that

10.7
$$\int_Y |K(x, y)| \, |f(y)| \, d\nu(y) \leq \Big(\int_Y |K(x, y)|^2 \, d\nu(y) \Big)^{1/2} \|f\|_{L^2(\nu)}.$$

for every $x \in X$. Squaring both sides of the inequality above and then integrating on X with respect to μ gives

$$\begin{aligned} \int_X \Big(\int_Y |K(x, y)| \, |f(y)| \, d\nu(y) \Big)^2 d\mu(x) &\leq \Big(\int_X \int_Y |K(x, y)|^2 \, d\nu(y) \, d\mu(x) \Big) \|f\|_{L^2(\nu)}^2 \\ &= \|K\|_{L^2(\mu \times \nu)}^2 \|f\|_{L^2(\nu)}^2, \end{aligned}$$

where the last line holds by Tonelli's Theorem (5.28). The inequality above implies that the integral on the left side of 10.7 is finite for μ-almost every $x \in X$. Thus the integral in 10.6 makes sense for μ-almost every $x \in X$. Now the last inequality above shows that

$$\|\mathcal{I}_K f\|_{L^2(\mu)}^2 = \int_X |(\mathcal{I}_K f)(x)|^2 \, d\mu(x) \leq \|K\|_{L^2(\mu \times \nu)}^2 \|f\|_{L^2(\nu)}^2.$$

Thus $\mathcal{I}_K$ is a bounded linear map from $L^2(\nu)$ to $L^2(\mu)$ and

10.8
$$\|\mathcal{I}_K\| \leq \|K\|_{L^2(\mu \times \nu)}.$$

Define $K^* \colon Y \times X \to \mathbf{F}$ by

$$K^*(y, x) = \overline{K(x, y)},$$

and note that $K^* \in L^2(\nu \times \mu)$. Thus $\mathcal{I}_{K^*} \colon L^2(\mu) \to L^2(\nu)$ is a bounded linear map. Using Tonelli's Theorem (5.28) and Fubini's Theorem (5.32), we have

$$\begin{aligned} \langle \mathcal{I}_K f, g \rangle &= \int_X \int_Y K(x, y) f(y) \, d\nu(y) \overline{g(x)} \, d\mu(x) \\ &= \int_Y f(y) \int_X K(x, y) \overline{g(x)} \, d\mu(x) \, d\nu(y) \\ &= \int_Y f(y) \overline{\mathcal{I}_{K^*} g(y)} \, d\nu(y) = \langle f, \mathcal{I}_{K^*} g \rangle \end{aligned}$$

for all $f \in L^2(\nu)$ and all $g \in L^2(\mu)$. Thus

10.9
$$(\mathcal{I}_K)^* = \mathcal{I}_{K^*}.$$

10.10 Example *linear maps induced by matrices*

As a special case of the previous example, suppose $m, n \in \mathbf{Z}^+$, μ is counting measure on $\{1, \ldots, m\}$, ν is counting measure on $\{1, \ldots, n\}$, and K is an m-by-n matrix with entry $K(i, j) \in \mathbf{F}$ in row i, column j. In this case, the linear map $\mathcal{I}_K \colon L^2(\nu) \to L^2(\mu)$ induced by integration is given by the equation

$$(\mathcal{I}_K f)(i) = \sum_{j=1}^{n} K(i, j) f(j)$$

for $f \in L^2(\nu)$. If we identify $L^2(\nu)$ and $L^2(\mu)$ with $\mathbf{F}^n$ and $\mathbf{F}^m$ and then think of elements of $\mathbf{F}^n$ and $\mathbf{F}^m$ as column vectors, then the equation above shows that the linear map $\mathcal{I}_K \colon \mathbf{F}^n \to \mathbf{F}^m$ is simply matrix multiplication by K.

In this setting, K^* is called the *conjugate transpose* of K because the n-by-m matrix K^* is obtained by interchanging the rows and the columns of K and then taking the complex conjugate of each entry.

The previous example now shows that

$$\|\mathcal{I}_K\| \leq \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} |K(i, j)|^2 \Big)^{1/2}.$$

Furthermore, the previous example shows that the adjoint of the linear map of multiplication by the matrix K is the linear map of multiplication by the conjugate transpose matrix K^*, a result that may be familiar to you from linear algebra.

If T is a bounded linear map from a Hilbert space V to a Hilbert space W, then the adjoint T^* has been defined as a function from W to V. We now show that the adjoint T^* is linear and bounded. Recall that $\mathcal{B}(V, W)$ denotes the Banach space of bounded linear maps from V to W.

10.11 ***T^* is a bounded linear map***

Suppose V and W are Hilbert spaces and $T \in \mathcal{B}(V, W)$. Then

$$T^* \in \mathcal{B}(W, V), \quad (T^*)^* = T, \quad \text{and} \quad \|T^*\| = \|T\|.$$

Proof Suppose $g_1, g_2 \in W$. Then

$$\begin{aligned} \langle f, T^*(g_1 + g_2) \rangle = \langle Tf, g_1 + g_2 \rangle &= \langle Tf, g_1 \rangle + \langle Tf, g_2 \rangle \\ &= \langle f, T^* g_1 \rangle + \langle f, T^* g_2 \rangle \\ &= \langle f, T^* g_1 + T^* g_2 \rangle \end{aligned}$$

for all $f \in V$. Thus $T^*(g_1 + g_2) = T^* g_1 + T^* g_2$.

Suppose $\alpha \in \mathbf{F}$ and $g \in W$. Then

$$\langle f, T^*(\alpha g) \rangle = \langle Tf, \alpha g \rangle = \overline{\alpha} \langle Tf, g \rangle = \overline{\alpha} \langle f, T^* g \rangle = \langle f, \alpha T^* g \rangle$$

for all $f \in V$. Thus $T^*(\alpha g) = \alpha T^* g$.

We have now shown that $T^* \colon W \to V$ is a linear map. From 10.3, we see that T^* is bounded. In other words, $T^* \in \mathcal{B}(W, V)$.

Because $T^* \in \mathcal{B}(W,V)$, its adjoint $(T^*)^* \colon V \to W$ is defined. Suppose $f \in V$. Then

$$\langle (T^*)^* f, g\rangle = \overline{\langle g, (T^*)^* f\rangle} = \overline{\langle T^* g, f\rangle} = \langle f, T^* g\rangle = \langle Tf, g\rangle$$

for all $g \in W$. Thus $(T^*)^* f = Tf$, and hence $(T^*)^* = T$.

From 10.3, we see that $\|T^*\| \leq \|T\|$. Applying this inequality with T replaced by T^* we have

$$\|T^*\| \leq \|T\| = \|(T^*)^*\| \leq \|T^*\|.$$

Because the first and last terms above are the same, the first inequality must be an equality. In other words, we have $\|T^*\| = \|T\|$. ■

Parts (a) and (b) of the next result show that if V and W are real Hilbert spaces, then the function $T \mapsto T^*$ from $\mathcal{B}(V,W)$ to $\mathcal{B}(W,V)$ is a linear map. However, if V and W are nonzero complex Hilbert spaces, then $T \mapsto T^*$ is not a linear map because of the complex conjugate in (b).

10.12 *properties of the adjoint*

Suppose V, W, and U are Hilbert spaces. Then

(a) $(S+T)^* = S^* + T^*$ for all $S, T \in \mathcal{B}(V,W)$;

(b) $(\alpha T)^* = \overline{\alpha} T^*$ for all $\alpha \in \mathbf{F}$ and all $T \in \mathcal{B}(V,W)$;

(c) $I^* = I$, where I is the identity operator on V;

(d) $(S \circ T)^* = T^* \circ S^*$ for all $T \in \mathcal{B}(V,W)$ and $S \in \mathcal{B}(W,U)$.

Proof

(a) The proof of (a) is left to the reader as an exercise.

(b) Suppose $\alpha \in \mathbf{F}$ and $T \in \mathcal{B}(V,W)$. If $f \in V$ and $g \in W$, then

$$\langle f, (\alpha T)^* g\rangle = \langle \alpha T f, g\rangle = \alpha \langle Tf, g\rangle = \alpha \langle f, T^* g\rangle = \langle f, \overline{\alpha} T^* g\rangle.$$

Thus $(\alpha T)^* g = \overline{\alpha} T^* g$, as desired.

(c) If $f, g \in V$, then

$$\langle f, I^* g\rangle = \langle If, g\rangle = \langle f, g\rangle.$$

Thus $I^* g = g$, as desired.

(d) Suppose $T \in \mathcal{B}(V,W)$ and $S \in \mathcal{B}(W,U)$. If $f \in V$ and $g \in U$, then

$$\langle f, (S \circ T)^* g\rangle = \langle (S \circ T) f, g\rangle = \langle S(Tf), g\rangle = \langle Tg, S^* g\rangle = \langle f, T^*(S^* g)\rangle.$$

Thus $(S \circ T)^* g = T^*(S^* g) = (T^* \circ S^*)(g)$. Hence $(S \circ T)^* = T^* \circ S^*$, as desired. ■

Null Spaces and Ranges in Terms of Adjoints

The next result shows the relationship between the null space and the range of a linear map and its adjoint. The orthogonal complement of each subset of a Hilbert space is closed [see 8.40(a)]. However, the range of a bounded linear map on a Hilbert space need not be closed (see Example 10.15 or Exercises 9 and 10 for examples). Thus in parts (b) and (d) of the result below, we must take the closure of the range.

10.13 ***null space and range of*** T^*

Suppose V and W are Hilbert spaces and $T \in \mathcal{B}(V, W)$. Then

(a) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$;

(b) $\overline{\operatorname{range} T^*} = (\operatorname{null} T)^\perp$;

(c) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$;

(d) $\overline{\operatorname{range} T} = (\operatorname{null} T^*)^\perp$.

Proof We begin by proving (a). Let $g \in W$. Then

$$\begin{aligned} g \in \operatorname{null} T^* &\iff T^* g = 0 \\ &\iff \langle f, T^* g \rangle = 0 \text{ for all } f \in V \\ &\iff \langle Tf, g \rangle = 0 \text{ for all } f \in V \\ &\iff g \in (\operatorname{range} T)^\perp. \end{aligned}$$

Thus $\operatorname{null} T^* = (\operatorname{range} T)^\perp$, proving (a).

If we take the orthogonal complement of both sides of (a), we get (d), where we have used 8.41. Replacing T with T^* in (a) gives (c), where we have used 10.11. Finally, replacing T with T^* in (d) gives (b). ■

As a corollary of the result above, we have the following result, which gives a useful way to determine whether or not a linear map has a dense range.

10.14 ***necessary and sufficient condition for dense range***

Suppose V and W are Hilbert spaces and $T \in \mathcal{B}(V, W)$. Then T has dense range if and only if T^* is injective.

Proof From 10.13(d) we see that T has dense range if and only if $(\operatorname{null} T^*)^\perp = W$, which happens if and only if $\operatorname{null} T^* = \{0\}$, which happens if and only if T^* is injective. ■

The advantage of using the result above is that to determine whether or not a bounded linear map T between Hilbert spaces has a dense range, we need only determine whether or not 0 is the only solution to the equation $T^* g = 0$. The next example illustrates this procedure.

10.15 Example *Volterra operator*

The *Volterra operator* is the linear map $\mathcal{V}\colon L^2([0,1]) \to L^2([0,1])$ defined by

$$(\mathcal{V}f)(x) = \int_0^x f(y)\,dy$$

for $f \in L^2([0,1])$ and $x \in [0,1]$; here dy means $d\lambda(y)$, where λ is the usual Lebesgue measure on the interval $[0,1]$.

To show that $\mathcal{V}$ is a bounded linear map from $L^2([0,1])$ to $L^2([0,1])$, let K be the function on $[0,1] \times [0,1]$ defined by

$$K(x,y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \le y. \end{cases}$$

In other words, K is the characteristic function of the triangle below the diagonal of the unit square. Clearly $K \in \mathcal{L}^2(\lambda \times \lambda)$ and $\mathcal{V} = \mathcal{I}_K$ as defined in 10.6. Thus $\mathcal{V}$ is a bounded linear map from $L^2([0,1])$ to $L^2([0,1])$ and $\|\mathcal{V}\| \le \frac{1}{\sqrt{2}}$ (by 10.8).

Vito Volterra (1860–1940) was a pioneer in developing functional analytic techniques to study integral equations.

Because $\mathcal{V}^* = \mathcal{I}_{K^*}$ (by 10.9) and K^* is the characteristic function of the closed triangle above the diagonal of the unit square, we see that

10.16 $$(\mathcal{V}^*f)(x) = \int_x^1 f(y)\,dy = \int_0^1 f(y)\,dy - \int_0^x f(y)\,dy$$

for $f \in L^2([0,1])$ and $x \in [0,1]$.

Now we can show that $\mathcal{V}^*$ is injective. To do this, suppose $f \in L^2([0,1])$ and $\mathcal{V}^*f = 0$. Differentiating both sides of 10.16 with respect to x and using the Lebesgue Differentiation Theorem (4.19), we conclude that $f = 0$. Hence $\mathcal{V}^*$ is injective. Thus the Volterra operator $\mathcal{V}$ has dense range (by 10.14).

Although range $\mathcal{V}$ is dense in $L^2([0,1])$, it does not equal $L^2([0,1])$ (because every element of range $\mathcal{V}$ is a continuous function on $[0,1]$ that vanishes at 0). Thus the Volterra operator $\mathcal{V}$ has dense but not closed range in $L^2([0,1])$.

Invertibility of Operators

Linear maps from a vector space to itself are so important that they get a special name and special notation.

10.17 **Definition** ***operator;*** $\mathcal{B}(V)$

- An *operator* is a linear map from a vector space to itself.
- If V is a normed vector space, then $\mathcal{B}(V)$ denotes the normed vector space of bounded operators on V. In other words, $\mathcal{B}(V) = \mathcal{B}(V,V)$.

10.18 Definition ***invertible;*** T^{-1}

- An operator T on a vector space V is called *invertible* if T is a one-to-one and surjective linear map of V onto V.
- Equivalently, an operator $T\colon V \to V$ is invertible if and only if there exists an operator $T^{-1}\colon V \to V$ such that $T^{-1} \circ T = T \circ T^{-1} = I$.

The second bullet point above is equivalent to the first bullet point because if a linear map $T\colon V \to V$ is one-to-one and surjective, then the inverse function $T^{-1}\colon V \to V$ is automatically linear (as you should verify).

Also, if V is a Banach space and T is a bounded operator on V that is invertible, then the inverse T^{-1} is automatically bounded, as follows from the Bounded Inverse Theorem (6.83).

The next result shows that inverses and adjoints work well together. In the proof, we use the common convention of writing composition of linear maps with the same notation as multiplication. In other words, if S and T are linear maps such that $S \circ T$ makes sense, then from now on

$$ST = S \circ T.$$

10.19 ***inverse of the adjoint equals adjoint of the inverse***

A bounded operator T on a Hilbert space is invertible if and only if T^* is invertible. Furthermore, if T is invertible, then $(T^*)^{-1} = (T^{-1})^*$.

Proof First suppose T is invertible. Taking the adjoint of all three sides of the equation $T^{-1}T = TT^{-1} = I$, we get

$$T^*(T^{-1})^* = (T^{-1})^*T^* = I,$$

which implies that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Now suppose T^* is invertible. Then by the direction just proved, $(T^*)^*$ is invertible. Because $(T^*)^* = T$, this implies that T is invertible, completing the proof. ■

Norms work well with the composition of linear maps, as shown in the next result.

10.20 ***norm of a composition of linear maps***

Suppose U, V, W are normed vector spaces, $T \in \mathcal{B}(U, V)$, and $S \in \mathcal{B}(V, W)$. Then

$$\|ST\| \le \|S\|\,\|T\|.$$

Proof If $f \in U$, then

$$\|(ST)(f)\| = \|S(Tf)\| \le \|S\|\,\|Tf\| \le \|S\|\,\|T\|\,\|f\|.$$

Thus $\|ST\| \le \|S\|\,\|T\|$, as desired. ■

Unlike linear maps from one vector space to a different vector space, operators on the same vector space can be composed with each other and raised to powers.

10.21 Definition T^k

Suppose T is an operator on a vector space V.

- For $k \in \mathbf{Z}^+$, the operator T^k is defined by $T^k = \underbrace{TT\cdots T}_{k \text{ times}}$.
- T^0 is defined to be the identity operator $I\colon V \to V$.

You should verify that powers of an operator satisfy the usual arithmetic rules: $T^jT^k = T^{j+k}$ and $(T^j)^k = T^{jk}$ for $j, k \in \mathbf{Z}^+$. Also, if V is a normed vector space and $T \in \mathcal{B}(V)$, then

$$\|T^k\| \leq \|T\|^k$$

for every $k \in \mathbf{Z}^+$, as follows from using induction on 10.20.

Recall that if $z \in \mathbf{C}$ with $|z| < 1$, then the formula for the sum of a geometric series shows that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

The next result shows that this formula carries over to operators on Banach spaces.

10.22 ***operators in the open unit ball centered at the identity are invertible***

If T is a bounded operator on a Banach space and $\|T\| < 1$, then $I - T$ is invertible and

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Proof Suppose T is a bounded operator on a Banach space V and $\|T\| < 1$. Then

$$\sum_{k=0}^{\infty} \|T^k\| \leq \sum_{k=0}^{\infty} \|T\|^k = \frac{1}{1-\|T\|} < \infty.$$

Hence 6.47 and 6.41 imply that the infinite sum $\sum_{k=0}^{\infty} T^k$ converges in $\mathcal{B}(V)$. Now

10.23 $$(I-T)\sum_{k=0}^{\infty} T^k = \lim_{n\to\infty}(I-T)\sum_{k=0}^{n} T^k = \lim_{n\to\infty}(I-T^{n+1}) = I,$$

where the last equality holds because $\|T^{n+1}\| \leq \|T\|^{n+1}$ and $\|T\| < 1$. Similarly,

10.24 $$\Big(\sum_{k=0}^{\infty} T^k\Big)(I-T) = \lim_{n\to\infty}\sum_{k=0}^{n} T^k(I-T) = \lim_{n\to\infty}(I-T^{n+1}) = I.$$

Equations 10.23 and 10.24 imply that $I - T$ is invertible and $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$. ∎

Now we use the previous result to show that the set of invertible operators on a Banach space is open.

10.25 ***invertible operators form an open set***

Suppose V is a Banach space. Then $\{T \in \mathcal{B}(V) : T \text{ is invertible}\}$ is an open subset of $\mathcal{B}(V)$.

Proof Suppose $T \in \mathcal{B}(V)$ is invertible. Suppose $S \in \mathcal{B}(V)$ and

$$\|T - S\| < \frac{1}{\|T^{-1}\|}.$$

Then

$$\|I - T^{-1}S\| = \|T^{-1}T - T^{-1}S\| \leq \|T^{-1}\| \, \|T - S\| < 1.$$

Hence 10.22 implies that $I - (I - T^{-1}S)$ is invertible; in other words, $T^{-1}S$ is invertible.

Now $S = T(T^{-1}S)$. Thus S is the product of two invertible operators, which implies that S is invertible with $S^{-1} = (T^{-1}S)^{-1}T^{-1}$.

We have shown that every element of the open ball of radius $\|T^{-1}\|^{-1}$ centered at T is invertible. Thus the set of invertible elements of $\mathcal{B}(V)$ is open. ∎

10.26 Definition ***left invertible; right invertible***

Suppose T is a bounded operator on a Banach space V.

- T is called *left invertible* if there exists $S \in \mathcal{B}(V)$ such that $ST = I$.
- T is called *right invertible* if there exists $S \in \mathcal{B}(V)$ such that $TS = I$.

One of the wonderful theorems of linear algebra states that left invertibility and right invertibility and invertibility are all equivalent to each other for operators on a finite-dimensional vector space. The next example shows that this result fails on infinite-dimensional Hilbert spaces.

10.27 Example *left invertibility is not equivalent to right invertibility*

Define the *right shift* $T \colon \ell^2 \to \ell^2$ and the *left shift* $S \colon \ell^2 \to \ell^2$ by

$$T(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots)$$

and

$$S(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots).$$

Because $ST = I$, we see that T is left invertible and S is right invertible. However, T is neither invertible nor right invertible because it is not surjective, and S is neither invertible nor left invertible because it is not injective.

The result 10.29 below gives equivalent conditions for an operator on a Hilbert space to be left invertible. On finite-dimensional vector spaces, left invertibility is equivalent to injectivity. The example below shows that this fails on infinite-dimensional Hilbert spaces. Thus we cannot eliminate the closed range requirement in part (c) of 10.29.

10.28 **Example** *injective but not left invertible*

Define $T\colon \ell^2 \to \ell^2$ by

$$T(a_1, a_2, a_3, \dots) = \Big(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots\Big).$$

Then T is an injective bounded operator on ℓ^2.

Suppose S is an operator on ℓ^2 such that $ST = I$. For $n \in \mathbf{Z}^+$, let $e_n \in \ell^2$ be the vector with 1 in the nth-slot and 0 elsewhere. Then

$$Se_n = S(nTe_n) = n(ST)(e_n) = ne_n.$$

The equation above implies that S is unbounded. Thus T is not left invertible, even though T is injective.

10.29 ***left invertibility***

Suppose V is a Hilbert space and $T \in \mathcal{B}(V)$. Then the following are equivalent:

(a) T is left invertible.

(b) there exists $\alpha \in (0, \infty)$ such that $\|f\| \le \alpha \|Tf\|$ for all $f \in V$.

(c) T is injective and has closed range.

(d) T^*T is invertible.

Proof First suppose (a) holds. Thus there exists $S \in \mathcal{B}(V)$ such that $ST = I$. If $f \in V$, then

$$\|f\| = \|S(Tf)\| \le \|S\|\,\|Tf\|.$$

Thus (b) holds with $\alpha = \|S\|$, proving that (a) implies (b).

Now suppose (b) holds. Thus there exists $\alpha \in (0, \infty)$ such that

10.30 $$\|f\| \le \alpha \|Tf\| \text{ for all } f \in V.$$

The inequality above shows that if $f \in V$ and $Tf = 0$, then $f = 0$. Thus T is injective. To show that T has closed range, suppose $f_1, f_2, \dots$ is a sequence in V such $Tf_1, Tf_2, \dots$ converges in V to some $g \in V$. Thus the sequence $Tf_1, Tf_2, \dots$ is a Cauchy sequence in V. The inequality 10.30 then implies that $f_1, f_2, \dots$ is a Cauchy sequence in V. Thus $f_1, f_2, \dots$ converges in V to some $f \in V$, which implies that $Tf = g$. Hence $g \in \operatorname{range} T$, completing the proof that T has closed range, and completing the proof that (b) implies (c).

Now suppose (c) holds, so T is injective and has closed range. We want to prove that (a) holds. Let $R\colon \text{range}\, T \to V$ be the inverse of the one-to-one linear function $f \mapsto Tf$ that maps V onto range T. Because range T is a closed subspace of V and thus is a Banach space [by 6.16(b)], the Bounded Inverse Theorem (6.83) implies that R is a bounded linear map. Let P denote the orthogonal projection of V onto the closed subspace range T. Define $S\colon V \to V$ by

$$Sg = R(Pg).$$

Then for each $g \in V$, we have

$$\|Sg\| = \|R(Pg)\| \leq \|R\|\,\|Pg\| \leq \|R\|\|g\|,$$

where the last inequality comes from 8.37(d). The inequality above implies that S is a bounded operator on V. If $f \in V$, then

$$S(Tf) = R\big(P(Tf)\big) = R(Tf) = f.$$

Thus $ST = I$, which means that T is left invertible, completing the proof that (c) implies (a).

At this stage of the proof we know that (a), (b), and (c) are equivalent. To prove that one of these implies (d), suppose (b) holds. Squaring the inequality in (b), we see that if $f \in V$, then

$$\|f\|^2 \leq \alpha^2\|Tf\|^2 = \alpha^2\langle T^*Tf, f\rangle \leq \alpha^2\|T^*Tf\|\,\|f\|,$$

which implies that

$$\|f\| \leq \alpha^2\|T^*Tf\|.$$

In other words, (b) holds with T replaced by T^*T (and α replaced by α^2). By the equivalence we already proved between (a) and (b), we conclude that T^*T is left invertible. Thus there exists $S \in \mathcal{B}(V)$ such that $S(T^*T) = I$. Taking adjoints of both sides of the last equation shows that $(T^*T)S^* = I$. Thus T^*T is also right invertible, which implies that T^*T is invertible. Thus (b) implies (d).

Finally, suppose (d) holds, so T^*T is invertible. Hence there exists $S \in \mathcal{B}(V)$ such that $I = S(T^*T) = (ST^*)T$. Thus T is left invertible, showing that (d) implies (a), completing the proof that (a), (b), (c), and (d) are equivalent. ∎

You may be familiar with the finite-dimensional result that right invertibility is equivalent to surjectivity. The next result shows that this equivalency also holds on infinite-dimensional Hilbert spaces.

10.31 *right invertibility*

Suppose V is a Hilbert space and $T \in \mathcal{B}(V)$. Then the following are equivalent:

(a) T is right invertible.

(b) T is surjective.

(c) TT^* is invertible.

Proof Taking adjoints shows that an operator is right invertible if and only if its adjoint is left invertible. Thus the equivalence of (a) and (c) in this result follows immediately from the equivalence of (a) and (d) in 10.29 applied to T^* instead of T.

Suppose (a) holds, so T is right invertible. Hence there exists $S \in \mathcal{B}(V)$ such that $TS = I$. Thus $T(Sf) = f$ for every $f \in V$, which implies that T is surjective, completing the proof that (a) implies (b).

To prove that (b) implies (a), suppose T is surjective. Define $R\colon (\operatorname{null} T)^\perp \to V$ by $R = T|_{(\operatorname{null} T)^\perp}$. Clearly R is injective because

$$\operatorname{null} R = (\operatorname{null} T)^\perp \cap (\operatorname{null} T) = \{0\}.$$

If $f \in V$, then $f = g + h$ for some $g \in \operatorname{null} T$ and some $h \in (\operatorname{null} T)^\perp$ (by 8.43); thus $Tf = Th = Rh$, which implies that $\operatorname{range} T = \operatorname{range} R$. Because T is surjective, this implies that $\operatorname{range} R = V$. In other words, R is a continuous injective linear map of $(\operatorname{null} T)^\perp$ onto V. The Bounded Inverse Theorem (6.83) now implies that $R^{-1}\colon V \to (\operatorname{null} T)^\perp$ is a bounded linear map on V. We have $TR^{-1} = I$. Thus T is right invertible, completing the proof that (b) implies (a). ■

EXERCISES 10A

1 Define $T\colon \ell^2 \to \ell^2$ by $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Find a formula for T^*.

2 Suppose V is a Hilbert space, U is a closed subspace of V, and $T\colon U \to V$ is defined by $Tf = f$. Describe the linear operator $T^*\colon V \to U$.

3 Suppose V and W are Hilbert spaces and $g \in V$, $h \in W$. Define $T \in \mathcal{B}(V, W)$ by $Tf = \langle f, g\rangle h$. Find a formula for T^*.

4 Suppose V and W are Hilbert spaces and $T \in \mathcal{B}(V, W)$ has finite-dimensional range. Prove that T^* also has finite-dimensional range.

5 Prove or give a counterexample: If V is a Hilbert space and $T\colon V \to V$ is a bounded linear map such that $\dim \operatorname{null} T < \infty$, then $\dim \operatorname{null} T^* < \infty$.

6 Suppose T is a bounded linear map from a Hilbert space V to a Hilbert space W. Prove that $\|T^*T\| = \|T\|^2$.
[*This formula for* $\|T^*T\|$ *leads to the important subject of* C^**-algebras.*]

7 Suppose V is a Hilbert space and $\operatorname{Inv}(V)$ is the set of invertible bounded operators on V. Think of $\operatorname{Inv}(V)$ as a metric space with the metric it inherits as a subset of $\mathcal{B}(V)$. Show that $T \mapsto T^{-1}$ is a continuous function from $\operatorname{Inv}(V)$ to $\operatorname{Inv}(V)$.

8 Suppose T is a bounded operator on a Hilbert space.

(a) Prove that T is left invertible if and only if T^* is right invertible.

(b) Prove that T is invertible if and only if T is both left and right invertible.

9 Suppose $b_1, b_2, \ldots$ is a bounded sequence in $\mathbf{F}$. Define a bounded linear map $T: \ell^2 \to \ell^2$ by

$$T(a_1, a_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots).$$

(a) Find a formula for T^*.

(b) Show that T is injective if and only if $b_k \neq 0$ for every $k \in \mathbf{Z}^+$.

(c) Show that T has dense range if and only if $b_k \neq 0$ for every $k \in \mathbf{Z}^+$.

(d) Show that T has closed range if and only if

$$\inf\{|b_k| : k \in \mathbf{Z}^+ \text{ and } b_k \neq 0\} > 0.$$

(e) Show that T is invertible if and only if

$$\inf\{|b_k| : k \in \mathbf{Z}^+\} > 0.$$

10 Suppose $h \in \mathcal{L}^\infty(\mathbf{R})$ and $M_h: L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is the bounded operator defined by $M_h f = fh$.

(a) Show that M_h is injective if and only if $|\{x \in \mathbf{R} : h(x) = 0\}| = 0$.

(b) Find a necessary and sufficient condition (in terms of h) for M_h to have dense range.

(c) Find a necessary and sufficient condition (in terms of h) for M_h to have closed range.

(d) Find a necessary and sufficient condition (in terms of h) for M_h to be invertible.

11 (a) Prove or give a counterexample: If T is a bounded operator on a Hilbert space such that T and T^* are both injective, then T is invertible.

(b) Prove or give a counterexample: If T is a bounded operator on a Hilbert space such that T and T^* are both surjective, then T is invertible.

12 Define $T: \ell^2 \to \ell^2$ by $T(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$. Suppose $\alpha \in \mathbf{F}$.

(a) Prove that $T - \alpha I$ is injective if and only if $|\alpha| \geq 1$.

(b) Prove that $T - \alpha I$ is invertible if and only if $|\alpha| > 1$.

(c) Prove that $T - \alpha I$ is surjective if and only if $|\alpha| \neq 1$.

(d) Prove that $T - \alpha I$ is left invertible if and only if $|\alpha| > 1$.

13 Suppose V is a Hilbert space.

(a) Show that $\{T \in \mathcal{B}(V) : T \text{ is left invertible}\}$ is an open subset of $\mathcal{B}(V)$.

(b) Show that $\{T \in \mathcal{B}(V) : T \text{ is right invertible}\}$ is an open subset of $\mathcal{B}(V)$.

14 Suppose T is a bounded operator on a Hilbert space V.

(a) Prove that T is invertible if and only if T has a unique left inverse. In other words, prove that T is invertible if and only if there exists a unique $S \in \mathcal{B}(V)$ such that $ST = I$.

(b) Prove that T is invertible if and only if T has a unique right inverse. In other words, prove that T is invertible if and only if there exists a unique $S \in \mathcal{B}(V)$ such that $TS = I$.

10B Spectrum

Spectrum of an Operator

The following definitions play key roles in operator theory.

10.32 Definition ***spectrum;*** $\mathrm{sp}(T)$***; eigenvalue***

Suppose T is a bounded operator on a Banach space V.

- A number $\alpha \in \mathbf{F}$ is called an *eigenvalue* of T if $T - \alpha I$ is not injective.
- A nonzero vector $f \in V$ is called an *eigenvector* of T corresponding to an eigenvalue $\alpha \in \mathbf{F}$ if
$$Tf = \alpha f.$$
- The *spectrum* of T is denoted $\mathrm{sp}(T)$ and is defined by
$$\mathrm{sp}(T) = \{\alpha \in \mathbf{F} : T - \alpha I \text{ is not invertible}\}.$$

If $T - \alpha I$ is not injective, then $T - \alpha I$ is not invertible. Thus the set of eigenvalues of a bounded operator T is contained in the spectrum of T. If V is a finite-dimensional Banach space and $T \in \mathcal{B}(V)$, then $T - \alpha I$ is not injective if and only if $T - \alpha I$ is not invertible. Thus if T is an operator on a finite-dimensional Banach space, then the spectrum of T equals the set of eigenvalues of T.

However, on infinite-dimensional Banach spaces, the spectrum of an operator does not necessarily equal the set of eigenvalues, as shown in the next example.

10.33 Example *eigenvalues and spectrum*

Verifying all the assertions in this example should help solidify your understanding of the definition of the spectrum.

- Suppose $b_1, b_2, \ldots$ is a bounded sequence in $\mathbf{F}$. Define a bounded linear map $T \colon \ell^2 \to \ell^2$ by
$$T(a_1, a_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots).$$
Then the set of eigenvalues of T equals $\{b_k : k \in \mathbf{Z}^+\}$ and the spectrum of T equals the closure of $\{b_k : k \in \mathbf{Z}^+\}$.
- Suppose $h \in \mathcal{L}^\infty(\mathbf{R})$. Define a bounded linear map $M_h \colon L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by
$$M_h f = fh.$$
Then $\alpha \in \mathbf{F}$ is an eigenvalue of M_h if and only if $|\{t \in \mathbf{R} : h(t) = \alpha\}| > 0$. Also, $\alpha \in \mathrm{sp}(M_h)$ if and only if $|\{t \in \mathbf{R} : |h(t) - \alpha| < \varepsilon\}| > 0$ for all $\varepsilon > 0$.
- Define the *right shift* $T \colon \ell^2 \to \ell^2$ and the *left shift* $S \colon \ell^2 \to \ell^2$ by
$$T(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots) \text{ and } S(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots).$$
Then T has no eigenvalues, and $\mathrm{sp}(T) = \{\alpha \in \mathbf{F} : |\alpha| \le 1\}$. Also, the set of eigenvalues of S is the open set $\{\alpha \in \mathbf{F} : |\alpha| < 1\}$, and the spectrum of S is the closed set $\{\alpha \in \mathbf{F} : |\alpha| \le 1\}$.

If α is an eigenvalue of an operator $T \in \mathcal{B}(V)$ and f is an eigenvector of T corresponding to α, then

$$\|Tf\| = \|\alpha f\| = |\alpha| \, \|f\|,$$

which implies that $|\alpha| \leq \|T\|$. The next result states that the same inequality holds for elements of $\operatorname{sp}(T)$.

10.34 $T - \alpha I$ *is invertible for* $|\alpha|$ *large*

Suppose T is a bounded operator on a Banach space. Then

(a) $\operatorname{sp}(T) \subset \{\alpha \in \mathbf{F} : |\alpha| \leq \|T\|\}$;

(b) $T - \alpha I$ is invertible for all $\alpha \in \mathbf{F}$ with $|\alpha| > \|T\|$;

(c) $\displaystyle\lim_{|\alpha| \to \infty} \|(T - \alpha I)^{-1}\| = 0$.

Proof We begin by proving (b). Suppose $\alpha \in \mathbf{F}$ and $|\alpha| > \|T\|$. Then

10.35
$$T - \alpha I = -\alpha\Big(I - \frac{T}{\alpha}\Big).$$

Because $\|T/\alpha\| < 1$, the equation above and 10.22 imply that $T - \alpha I$ is invertible, completing the proof of (b).

Using the definition of spectrum, (a) now follows immediately from (b).

To prove (c), again suppose $\alpha \in \mathbf{F}$ and $|\alpha| > \|T\|$. Then 10.35 and 10.22 imply

$$(T - \alpha I)^{-1} = -\frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{T^k}{\alpha^k}.$$

Thus

$$\begin{aligned} \|(T - \alpha I)^{-1}\| &\leq \frac{1}{|\alpha|} \sum_{k=0}^{\infty} \frac{\|T\|^k}{|\alpha|^k} \\ &= \frac{1}{|\alpha|} \frac{1}{1 - \frac{\|T\|}{|\alpha|}} \\ &= \frac{1}{|\alpha| - \|T\|}. \end{aligned}$$

The inequality above implies (c), completing the proof. ■

The set of eigenvalues of a bounded operator on a Hilbert space can be any bounded subset of $\mathbf{F}$, even a nonmeasurable set (see Exercise 3). In contrast, the next result shows that the spectrum of a bounded operator is a closed subset of $\mathbf{F}$. This result provides one indication that the spectrum of an operator may be a more useful set than the set of eigenvalues.

10.36 ***spectrum is closed***

The spectrum of a bounded operator on a Banach space is a closed subset of $\mathbf{F}$.

Proof Suppose T is a bounded operator on a Banach space V. Suppose $\alpha_1, \alpha_2, \ldots$ is a sequence in $\operatorname{sp}(T)$ that converges to some $\alpha \in \mathbf{F}$. Thus each $T - \alpha_n I$ is not invertible and

$$\lim_{n\to\infty}(T - \alpha_n I) = T - \alpha I.$$

The set of noninvertible elements of $\mathcal{B}(V)$ is a closed subset of $\mathcal{B}(V)$ (by 10.25). Hence the equation above implies that $T - \alpha I$ is not invertible. In other words, $\alpha \in \operatorname{sp}(T)$, which implies that $\operatorname{sp}(T)$ is closed. ■

Our next result provides the key tool used in proving that the spectrum of a bounded operator on a nonzero complex Hilbert space is nonempty (see 10.38). The statement of the next result and the proofs of the next two results use a bit of basic complex analysis. Because $\operatorname{sp}(T)$ is a closed subset of $\mathbf{C}$ (by 10.36), $\mathbf{C} \setminus \operatorname{sp}(T)$ is an open subset of $\mathbf{C}$ and thus it makes sense to ask whether the function in the result below is analytic.

To keep things simple, the next two results are stated for complex Hilbert spaces. See Exercise 6 for the analogous results for complex Banach spaces.

10.37 ***analyticity of*** $(T - \alpha I)^{-1}$

Suppose T is a bounded operator on a complex Hilbert space V. Then the function

$$\alpha \mapsto \langle (T - \alpha I)^{-1} f, g \rangle$$

is analytic on $\mathbf{C} \setminus \operatorname{sp}(T)$ for every $f, g \in V$.

Proof Suppose $\beta \in \mathbf{C} \setminus \operatorname{sp}(T)$. Then for $\alpha \in \mathbf{C}$ with $|\alpha - \beta| < 1/\|(T - \beta I)^{-1}\|$, we see from 10.22 that $I - (\alpha - \beta)(T - \beta I)^{-1}$ is invertible and

$$\bigl(I - (\alpha - \beta)(T - \beta I)^{-1}\bigr)^{-1} = \sum_{k=0}^{\infty} (\alpha - \beta)^k \bigl((T - \beta I)^{-1}\bigr)^k.$$

Multiplying both sides of the equation above by $(T - \beta I)^{-1}$ and using the equation $A^{-1}B^{-1} = (BA)^{-1}$ for invertible operators A and B, we get

$$(T - \alpha I)^{-1} = \sum_{k=0}^{\infty} (\alpha - \beta)^k \bigl((T - \beta I)^{-1}\bigr)^{k+1}.$$

Thus for $f, g \in V$, we have

$$\langle (T - \alpha I)^{-1} f, g \rangle = \sum_{k=0}^{\infty} \Bigl\langle \bigl((T - \beta I)^{-1}\bigr)^{k+1} f, g \Bigr\rangle (\alpha - \beta)^k.$$

The equation above shows that the function $\alpha \mapsto \langle (T - \alpha I)^{-1} f, g \rangle$ has a power series expansion as powers of $\alpha - \beta$ for α near β. Thus this function is analytic near β. ■

A major result in finite-dimensional linear algebra states that every operator on a nonzero finite-dimensional complex vector space has an eigenvalue. We have seen examples showing that this result does not extend to bounded operators on complex Hilbert spaces. However, the next result is an excellent substitute. Although a bounded operator on a nonzero complex Hilbert space need not have an eigenvalue, the next result shows that for each such operator T, there exists $\alpha \in \mathbf{C}$ such that $T - \alpha I$ is not invertible.

The spectrum of a bounded operator on a nonzero real Hilbert space can be the empty set. This can happen even in finite dimensions, where an operator on $\mathbf{R}^2$ might have no eigenvalues. Thus the restriction in the next result to the complex case cannot be removed.

10.38 *spectrum is nonempty*

The spectrum of a bounded operator on a complex nonzero Hilbert space is a nonempty subset of $\mathbf{C}$.

Proof Suppose $T \in \mathcal{B}(V)$, where V is a complex Hilbert space with $V \neq \{0\}$, and $\operatorname{sp}(T) = \emptyset$. Let $f \in V$ with $f \neq 0$. Take $g = T^{-1}f$ in 10.37. Because $\operatorname{sp}(T) = \emptyset$, 10.37 implies that the function

$$\alpha \mapsto \langle (T - \alpha I)^{-1} f, T^{-1} f \rangle$$

is analytic on all of $\mathbf{C}$. The value of the function above at $\alpha = 0$ equals the average value of the function on each circle in $\mathbf{C}$ centered at 0 (because analytic functions satisfy the mean value property). But 10.34(c) implies that this function has limit 0 as $|\alpha| \to \infty$. Thus taking the average over large circles, we see that the value of the function above at $\alpha = 0$ is 0. In other words,

$$\langle T^{-1} f, T^{-1} f \rangle = 0.$$

Hence $T^{-1}f = 0$. Applying T to both sides of the equation $T^{-1}f = 0$ shows that $f = 0$, which contradicts our assumption that $f \neq 0$. This contradiction means that our assumption that $\operatorname{sp}(T) = \emptyset$ was false, completing the proof. ∎

10.39 Definition $p(T)$

Suppose T is an operator on a vector space V and p is a polynomial with coefficients in $\mathbf{F}$:

$$p(z) = b_0 + b_1 z + \cdots + b_n z^n.$$

Then $p(T)$ is the operator on V defined by

$$p(T) = b_0 I + b_1 T + \cdots + b_n T^n.$$

You should verify that if p and q are polynomials with coefficients in $\mathbf{F}$ and T is an operator, then

$$(pq)(T) = p(T)\,q(T).$$

The next result provides a nice way to compute the spectrum of a polynomial applied to an operator. For example, this result implies that if T is a bounded operator on a complex Banach space, then the spectrum of T^2 consists of the squares of all numbers in the spectrum of T.

As with the previous result, the next result fails on real Banach spaces. As you can see, the proof below uses factorization of a polynomial with complex coefficients as the product of polynomials with degree 1, which is not necessarily possible when restricting to the field of real numbers.

10.40 ***Spectral Mapping Theorem***

Suppose T is a bounded operator on a complex Banach space and p is a polynomial with complex coefficients. Then

$$\operatorname{sp}\bigl(p(T)\bigr) = p\bigl(\operatorname{sp}(T)\bigr).$$

Proof If p is a constant polynomial, then both sides of the equation above consist of the set containing just that constant. Thus we can assume that p is a nonconstant polynomial.

First suppose $\alpha \in \operatorname{sp}\bigl(p(T)\bigr)$. Thus $p(T) - \alpha I$ is not invertible. By the Fundamental Theorem of Algebra, there exist $c, \beta_1, \dots \beta_n \in \mathbf{C}$ with $c \neq 0$ such that

10.41 $$p(z) - \alpha = c(z - \beta_1) \cdots (z - \beta_n)$$

for all $z \in \mathbf{C}$. Thus

$$p(T) - \alpha I = c(T - \beta_1 I) \cdots (T - \beta_n I).$$

The left side of the equation above is not invertible. Hence $T - \beta_k I$ is not invertible for some $k \in \{1, \dots, n\}$. Thus $\beta_k \in \operatorname{sp}(T)$. Now 10.41 implies $p(\beta_k) = \alpha$. Hence $\alpha \in p\bigl(\operatorname{sp}(T)\bigr)$, completing the proof that $\operatorname{sp}\bigl(p(T)\bigr) \subset p\bigl(\operatorname{sp}(T)\bigr)$.

To prove the inclusion in the other direction, now suppose $\beta \in \operatorname{sp}(T)$. The polynomial $z \mapsto p(z) - p(\beta)$ has a zero at β. Hence there exists a polynomial q with degree 1 less than the degree of p such that

$$p(z) - p(\beta) = (z - \beta)q(z)$$

for all $z \in \mathbf{C}$. Thus

10.42 $$p(T) - p(\beta)I = (T - \beta I)q(T)$$

and

10.43 $$p(T) - p(\beta)I = q(T)(T - \beta I).$$

Because $T - \beta I$ is not invertible, $T - \beta I$ is not surjective or $T - \beta I$ is not injective. If $T - \beta I$ is not surjective, then 10.42 shows that $p(T) - p(\beta)I$ is not surjective. If $T - \beta I$ is not injective, then 10.43 shows that $p(T) - p(\beta)I$ is not injective. Either way, we see that $p(T) - p(\beta)I$ is not invertible. Thus $p(\beta) \in \operatorname{sp}\bigl(p(T)\bigr)$, completing the proof that $\operatorname{sp}\bigl(p(T)\bigr) \supset p\bigl(\operatorname{sp}(T)\bigr)$. ∎

Self-adjoint Operators

In this subsection, we look at a nice special class of bounded operators.

10.44 Definition ***self-adjoint***

A bounded operator T on a Hilbert space is called *self-adjoint* if $T^* = T$.

The definition of the adjoint implies that a bounded operator T on a Hilbert space V is self-adjoint if and only if $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in V$. See Exercise 7 for an interesting result regarding this last condition.

10.45 Example *self-adjoint operators*

- Suppose $b_1, b_2, \ldots$ is a bounded sequence in $\mathbf{F}$. Define a bounded operator $T\colon \ell^2 \to \ell^2$ by
$$T(a_1, a_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots).$$
Then $T^*\colon \ell^2 \to \ell^2$ is the operator defined by
$$T^*(a_1, a_2, \ldots) = (a_1 \overline{b_1}, a_2 \overline{b_2}, \ldots).$$
Hence T is self-adjoint if and only if $b_k \in \mathbf{R}$ for all $k \in \mathbf{Z}^+$.

- More generally, suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $h \in \mathcal{L}^\infty(\mu)$. Define a bounded operator $M_h \in \mathcal{B}\big(L^2(\mu)\big)$ by $M_h f = fh$. Then $M_h{}^* = M_{\overline{h}}$. Thus M_h is self-adjoint if and only if $\mu(\{x \in X : h(x) \notin \mathbf{R}\}) = 0$.

- Suppose $n \in \mathbf{Z}^+$, K is an n-by-n matrix, and $\mathcal{I}_K\colon \mathbf{F}^n \to \mathbf{F}^n$ is the operator of matrix multiplication by K (thinking of elements of $\mathbf{F}^n$ as column vectors). Then $(\mathcal{I}_K)^*$ is the operator of multiplication by the conjugate transpose of K, as shown in Example 10.10. Thus $\mathcal{I}_K$ is a self-adjoint operator if and only if the matrix K equals its conjugate transpose.

- More generally, suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space, $K \in \mathcal{L}^2(\mu \times \mu)$, and $\mathcal{I}_K$ is the integral operator on $L^2(\mu)$ defined in Example 10.5. Define $K^*\colon X \times X \to \mathbf{F}$ by $K^*(y, x) = \overline{K(x, y)}$. Then $(\mathcal{I}_K)^*$ is the integral operator induced by K^*, as shown in Example 10.5. Thus if $K^* = K$, or in other words if $K(x, y) = \overline{K(y, x)}$ for all $(x, y) \in X \times X$, then $\mathcal{I}_K$ is self-adjoint.

- Suppose U is a closed subspace of a Hilbert space V. Recall that P_U denotes the orthogonal projection of V onto U (see Section 8B). We have
$$\begin{aligned}\langle P_U f, g\rangle &= \langle P_U f, P_U g + (I - P_U)g\rangle \\ &= \langle P_U f, P_U g\rangle \\ &= \langle f - (I - P_U)f, P_U g\rangle \\ &= \langle f, P_U g\rangle,\end{aligned}$$
where the second and fourth equalities above hold because of 8.37(a). The equation above shows that P_U is a self-adjoint operator.

For real Hilbert spaces, the next result requires the additional hypothesis that T is self-adjoint. To see that this extra hypothesis cannot be eliminated, consider the operator $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x,y) = (-y,x)$. Then, $T \neq 0$, but with the standard inner product on $\mathbf{R}^2$, we have $\langle Tf, f\rangle = 0$ for all $f \in \mathbf{R}^2$ (which you can verify either algebraically or by thinking of T as counterclockwise rotation by a right angle).

10.46 $\langle Tf, f\rangle = 0$ *for all* f *implies* $T = 0$

Suppose V is a Hilbert space, $T \in \mathcal{B}(V)$, and $\langle Tf, f\rangle = 0$ for all $f \in V$.

(a) If $\mathbf{F} = \mathbf{C}$, then $T = 0$.

(b) If $\mathbf{F} = \mathbf{R}$ and T is self-adjoint, then $T = 0$.

Proof First suppose $\mathbf{F} = \mathbf{C}$. If $g, h \in V$, then

$$\langle Tg, h\rangle = \frac{\langle T(g+h), g+h\rangle - \langle T(g-h), g-h\rangle}{4} + \frac{\langle T(g+ih), g+ih\rangle - \langle T(g-ih), g-ih\rangle}{4}\, i,$$

as can be verified by computing the right side. Our hypothesis that $\langle Tf, f\rangle = 0$ for all $f \in V$ implies that the right side above equals 0. Thus $\langle Tg, h\rangle = 0$ for all $g, h \in V$. Taking $h = Tg$, we can conclude that $T = 0$, which completes the proof of (a).

Now suppose $\mathbf{F} = \mathbf{R}$ and T is self-adjoint. Then

10.47 $$\langle Tg, h\rangle = \frac{\langle T(g+h), g+h\rangle - \langle T(g-h), g-h\rangle}{4};$$

this is proved by computing the right side using the equation

$$\langle Th, g\rangle = \langle h, Tg\rangle = \langle Tg, h\rangle,$$

where the first equality holds because T is self-adjoint and the second equality holds because we are working in a real Hilbert space. Each term on the right side of 10.47 is of the form $\langle Tf, f\rangle$ for appropriate f. Thus $\langle Tg, h\rangle = 0$ for all $g, h \in V$. This implies that $T = 0$ (take $h = Tg$), completing the proof of (b). ■

Some insight into the adjoint can be obtained by thinking of the operation $T \mapsto T^*$ on $\mathcal{B}(V)$ as analogous to the operation $z \mapsto \overline{z}$ on $\mathbf{C}$. Under this analogy, the self-adjoint operators (characterized by $T^* = T$) correspond to the real numbers (characterized by $\overline{z} = z$). The first two bullet points in Example 10.45 illustrate this analogy, as we saw that a multiplication operator on $L^2(\mu)$ is self-adjoint if and only if the multiplier is real-valued almost everywhere.

The next two results deepen the analogy between the self-adjoint operators and the real numbers. First we see this analogy reflected in the behavior of $\langle Tf, f\rangle$, and then we see this analogy reflected in the spectrum of T.

10.48 *self-adjoint characterized by $\langle Tf, f\rangle$*

Suppose T is a bounded operator on a complex Hilbert space V. Then T is self-adjoint if and only if

$$\langle Tf, f\rangle \in \mathbf{R}$$

for all $f \in V$.

Proof Let $f \in V$. Then

$$\langle Tf, f\rangle - \overline{\langle Tf, f\rangle} = \langle Tf, f\rangle - \langle f, Tf\rangle = \langle Tf, f\rangle - \langle T^* f, f\rangle = \langle (T - T^*) f, f\rangle.$$

If $\langle Tf, f\rangle \in \mathbf{R}$ for every $f \in V$, then the left side of the equation above equals 0, so $\langle (T - T^*) f, f\rangle = 0$ for every $f \in V$. This implies that $T - T^* = 0$ [by 10.46(a)]. Hence T is self-adjoint.

Conversely, if T is self-adjoint, then the right side of the equation above equals 0, so $\langle Tf, f\rangle = \overline{\langle Tf, f\rangle}$ for every $f \in V$. This implies that $\langle Tf, f\rangle \in \mathbf{R}$ for every $f \in V$, as desired. ■

10.49 *self-adjoint operators have real spectrum*

Suppose T is a bounded self-adjoint operator on a Hilbert space. Then $\operatorname{sp}(T) \subset \mathbf{R}$.

Proof The desired result holds if $\mathbf{F} = \mathbf{R}$ because the spectrum of every operator on a real Hilbert space is, by definition, contained in $\mathbf{R}$.

Thus we assume that T is a bounded operator on a complex Hilbert space V. Suppose $\alpha, \beta \in \mathbf{R}$, with $\beta \neq 0$. If $f \in V$, then

$$\begin{aligned} \|\big(T - (\alpha + \beta i)I\big) f\| \, \|f\| &\geq \big|\langle \big(T - (\alpha + \beta i)I\big) f, f\rangle\big| \\ &= \big|\langle Tf, f\rangle - \alpha\|f\|^2 - \beta\|f\|^2 i\big| \\ &\geq |\beta| \, \|f\|^2, \end{aligned}$$

where the first inequality comes from the Cauchy–Schwarz inequality (8.11) and the last inequality holds because $\langle Tf, f\rangle - \alpha\|f\|^2 \in \mathbf{R}$ (by 10.48).

The inequality above implies that

$$\|f\| \leq \frac{1}{|\beta|} \|\big(T - (\alpha + \beta i)I\big) f\|$$

for all $f \in V$. Now the equivalence of (a) and (b) in 10.29 shows that $T - (\alpha + \beta i)I$ is left invertible.

Because T is self-adjoint, the adjoint of $T - (\alpha + \beta i)I$ is $T - (\alpha - \beta i)I$, which is left invertible by the same argument as above (just replace β by $-\beta$). Hence $T - (\alpha + \beta i)I$ is right invertible (because its adjoint is left invertible). Because the operator $T - (\alpha + \beta i)I$ is both left and right invertible, it is invertible. In other words, $\alpha + \beta i \notin \operatorname{sp}(T)$. Thus $\operatorname{sp}(T) \subset \mathbf{R}$, as desired. ■

We showed that a bounded operator on a complex nonzero Hilbert space has a nonempty spectrum. That result can fail on real Hilbert spaces (where by definition the spectrum is contained in **R**). For example, the operator T on $\mathbf{R}^2$ defined by $T(x, y) = (-y, x)$ has empty spectrum. However, the previous result and 10.38 can be used to show that every ***self-adjoint*** operator on a nonzero real Hilbert space has nonempty spectrum (see Exercise 9 for the details).

Although the spectrum of every self-adjoint operator is nonempty, it is not true that every self-adjoint operator has an eigenvalue. For example, the self-adjoint operator $M_x \in \mathcal{B}\big(L^2([0, 1])\big)$ defined by $(M_x f)(x) = x f(x)$ has no eigenvalues.

Normal Operators

Now we consider another nice special class of operators.

10.50 Definition ***normal operator***

A bounded operator T on a Hilbert space is called *normal* if it commutes with its adjoint. In other words, T is *normal* if

$$T^*T = TT^*.$$

Clearly every self-adjoint operator is normal, but there exist normal operators that are not self-adjoint, as shown in the next example.

10.51 Example *normal operators*

- Suppose μ is a positive measure, $h \in \mathcal{L}^\infty(\mu)$, and $M_h \in \mathcal{B}\big(L^2(\mu)\big)$ is the multiplication operator defined by $M_h f = fh$. Then ${M_h}^* = M_{\overline{h}}$, which means that M_h is self-adjoint if h is real valued. If $\mathbf{F} = \mathbf{C}$, then h can be complex valued and M_h is not necessarily self-adjoint. However,
$$ {M_h}^* M_h = M_{|h|^2} = M_h {M_h}^* $$
and thus M_h is a normal operator even when h is complex valued.

- Suppose T is the operator on $\mathbf{F}^2$ whose matrix with respect to the standard basis is
$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$
Then T is not self-adjoint because the matrix above is not equal to its conjugate transpose. However, $T^*T = 13I$ and $TT^* = 13I$, as you should verify. Because $T^*T = TT^*$, we conclude that T is a normal operator.

10.52 Example *an operator that is not normal*

Suppose T is the right shift on ℓ^2; thus $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Then T^* is the left shift: $T^*(a_1, a_2, \dots) = (a_2, a_3, \dots)$. Hence T^*T is the identity operator on ℓ^2 and TT^* is the operator $(a_1, a_2, a_3, \dots) \mapsto (0, a_2, a_3, \dots)$. Thus $T^*T \neq TT^*$, which means that T is not a normal operator.

10.53 *normal in terms of norms*

Suppose T is a bounded operator on a Hilbert space V. Then T is normal if and only if

$$\|Tf\| = \|T^*f\|$$

for all $f \in V$.

Proof If $f \in V$, then

$$\|Tf\|^2 - \|T^*f\|^2 = \langle Tf, Tf\rangle - \langle T^*f, T^*f\rangle = \langle (T^*T - TT^*)f, f\rangle.$$

If T is normal, then the right side of the equation above equals 0, which implies that the left side also equals 0 and hence $\|Tf\| = \|T^*f\|$.

Conversely, suppose $\|Tf\| = \|T^*f\|$ for all $f \in V$. Then the left side of the equation above equals 0, which implies that the right side also equals 0 for all $f \in V$. Because $T^*T - TT^*$ is self-adjoint, 10.46 now implies that $T^*T - TT^* = 0$. Thus T is normal, completing the proof. ■

Each complex number can be written in the form $a + bi$, where a and b are real numbers. Part (a) of the next result gives the analogous result for bounded operators on a complex Hilbert space, with self-adjoint operators playing the role of real numbers. We could call the operators A and B in part (a) the real and imaginary parts of the operator T. Part (b) below shows that normality depends upon whether these real and imaginary parts commute.

10.54 *operator is normal if and only if its real and imaginary parts commute*

Suppose T is a bounded operator on a complex Hilbert space V.

(a) There exist unique self-adjoint operators A, B on V such that $T = A + iB$.

(b) T is normal if and only if $AB = BA$, where A, B are as in part (a).

Proof Suppose $T = A + iB$, where A and B are self-adjoint. Then $T^* = A - iB$. Adding these equations for T and T^* and then dividing by 2 produces a formula for A; subtracting the equation for T^* from the equation for T and then dividing by $2i$ produces a formula for B. Specifically, we have

$$A = \frac{T + T^*}{2} \quad \text{and} \quad B = \frac{T - T^*}{2i},$$

which proves the uniqueness part of (a). The existence part of (a) is proved by defining A and B by the equations above and noting that A and B as defined above are self-adjoint and $T = A + iB$.

To prove (b), verify that if A and B are defined as in the equations above, then

$$AB - BA = \frac{T^*T - TT^*}{2i}.$$

Thus $AB = BA$ if and only if T is normal. ■

An operator on a finite-dimensional vector space is left invertible if and only if it is right invertible. We have seen that this result fails for bounded operators on infinite-dimensional Hilbert spaces. However, the next result shows that we recover this equivalency for normal operators.

10.55 *invertibility for normal operators*

Suppose V is a Hilbert space and $T \in \mathcal{B}(V)$ is normal. Then the following are equivalent:

(a) T is invertible.

(b) T is left invertible.

(c) T is right invertible.

(d) T is surjective.

(e) T is injective and has closed range.

(f) T^*T is invertible.

(g) TT^* is invertible.

Proof Because T is normal, (f) and (g) are clearly equivalent. From 10.29, we know that (f), (b), and (e) are equivalent to each other. From 10.31, we know that (g), (c), and (d) are equivalent to each other. Thus (b), (c), (d), (e), (f), and (g) are all equivalent to each other.

Clearly (a) implies (b).

Suppose (b) holds. We already know that (b) and (c) are equivalent; thus T is left invertible and T is right invertible. Hence T is invertible, proving that (b) implies (a) and completing the proof that (a) through (g) are all equivalent to each other. ■

The next result shows that a normal operator and its adjoint have the same eigenvectors, with eigenvalues that are complex conjugates of each other. This result can fail for operators that are not normal. For example, 0 is an eigenvalue of the left shift on ℓ^2 but its adjoint the right shift has no eigenvectors and no eigenvalues.

10.56 *T normal and $Tf = \alpha f$ implies $T^*f = \overline{\alpha} f$*

Suppose T is a normal operator on a Hilbert space V, $\alpha \in \mathbf{F}$, and $f \in V$. Then α is an eigenvalue of T with eigenvector f if and only if $\overline{\alpha}$ is an eigenvalue of T^* with eigenvector f.

Proof Because $(T - \alpha I)^* = T^* - \overline{\alpha} I$ and T is normal, $T - \alpha I$ commutes with its adjoint. Thus $T - \alpha I$ is normal. Hence 10.53 implies that

$$\|(T - \alpha I)f\| = \|(T^* - \overline{\alpha} I)f\|.$$

Thus $(T - \alpha I)f = 0$ if and only if $(T^* - \overline{\alpha} I)f = 0$, as desired. ■

Because every self-adjoint operator is normal, the following result also holds for self-adjoint operators.

10.57 ***orthogonal eigenvectors for normal operators***

Eigenvectors of a normal operator corresponding to distinct eigenvalues are orthogonal.

Proof Suppose α and β are distinct eigenvalues of a normal operator T, with corresponding eigenvectors f and g. Then 10.56 implies that $T^* f = \overline{\alpha} f$. Thus

$$(\beta - \alpha)\langle g, f\rangle = \langle \beta g, f\rangle - \langle g, \overline{\alpha} f\rangle = \langle Tg, f\rangle - \langle g, T^* f\rangle = 0.$$

Because $\alpha \neq \beta$, the equation above implies that $\langle g, f\rangle = 0$, as desired. ■

Isometries and Unitary Operators

10.58 Definition ***isometry; unitary operator***

Suppose T is a bounded operator on a Hilbert space V.

- T is called an *isometry* if $\|Tf\| = \|f\|$ for every $f \in V$.
- T is called *unitary* if $T^*T = TT^* = I$.

10.59 Example *isometries and unitary operators*

- Suppose $T \in \mathcal{B}(\ell^2)$ is the right shift defined by

  $$T(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

  Then T is an isometry but is not a unitary operator because $TT^* \neq I$ (as is clear without even computing T^* because T is not surjective).

- Suppose $T \in \mathcal{B}\bigl(\ell^2(\mathbf{Z})\bigr)$ is the right shift defined by

  $$(Tf)(n) = f(n-1)$$

  for $f\colon \mathbf{Z} \to \mathbf{F}$ with $\sum_{k=-\infty}^{\infty} |f(k)|^2 < \infty$. Then T is an isometry and is unitary.

- Suppose $b_1, b_2, \dots$ is a bounded sequence in $\mathbf{F}$. Define $T \in \mathcal{B}(\ell^2)$ by

  $$T(a_1, a_2, \dots) = (a_1 b_1, a_2 b_2, \dots).$$

  Then T is an isometry if and only if T is unitary if and only if $|b_k| = 1$ for all $k \in \mathbf{Z}^+$.

- More generally, suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $h \in \mathcal{L}^\infty(\mu)$. Define $M_h \in \mathcal{B}\bigl(L^2(\mu)\bigr)$ by $M_h f = fh$. Then T is an isometry if and only if T is unitary if and only if $\mu(\{x \in X : |h(x)| \neq 1\}) = 0$.

By definition, isometries preserve norms. The equivalence of (a) and (b) in the following result shows that isometries also preserve inner products.

10.60 *isometries preserve inner products*

Suppose T is a bounded operator on a Hilbert space V. Then the following are equivalent:

(a) T is an isometry.

(b) $\langle Tf, Tg\rangle = \langle f, g\rangle$ for all $f, g \in V$.

(c) $T^*T = I$.

(d) $\{Te_k\}_{k\in\Gamma}$ is an orthonormal family for every orthonormal family $\{e_k\}_{k\in\Gamma}$ in V.

(e) $\{Te_k\}_{k\in\Gamma}$ is an orthonormal family for some orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V.

Proof If $f \in V$, then

$$\|Tf\|^2 - \|f\|^2 = \langle Tf, Tf\rangle - \langle f, f\rangle = \langle (T^*T - I)f, f\rangle.$$

Thus $\|Tf\| = \|f\|$ for all $f \in V$ if and only if the right side of the equation above is 0 for all $f \in V$. Because $T^*T - I$ is self-adjoint, this happens if and only if $T^*T - I = 0$ (by 10.46). Thus (a) is equivalent to (c).

If $T^*T = I$, then $\langle Tf, Tg\rangle = \langle T^*Tf, g\rangle = \langle f, g\rangle$ for all $f, g \in V$. Thus (c) implies (b).

Taking $g = f$ in (b), we see that (b) implies (a). Hence we now know that (a), (b), and (c) are equivalent to each other.

To prove that (b) implies (d), suppose (b) holds. If $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in V, then $\langle Te_j, Te_k\rangle = \langle e_j, e_k\rangle$ for all $j, k \in \Gamma$, and thus $\{Te_k\}_{k\in\Gamma}$ is an orthonormal family in V. Hence (b) implies (d).

Because V has an orthonormal basis (see 8.67 or 8.75), (d) implies (e).

Finally, suppose (e) holds. Thus $\{Te_k\}_{k\in\Gamma}$ is an orthonormal family for some orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V. Suppose $f \in V$. Then by 8.63(a) we have

$$f = \sum_{j\in\Gamma} \langle f, e_j\rangle e_j,$$

which implies that

$$T^*Tf = \sum_{j\in\Gamma} \langle f, e_j\rangle T^*Te_j.$$

Thus if $k \in \Gamma$, then

$$\langle T^*Tf, e_k\rangle = \sum_{j\in\Gamma} \langle f, e_j\rangle \langle T^*Te_j, e_k\rangle = \sum_{j\in\Gamma} \langle f, e_j\rangle \langle Te_j, Te_k\rangle = \langle f, e_k\rangle,$$

where the last equality holds because $\langle Te_j, Te_k\rangle$ equals 1 if $j = k$ and equals 0 otherwise. Because the equality above holds for every e_k in the orthonormal basis $\{e_k\}_{k\in\Gamma}$, we conclude that $T^*Tf = f$. Thus (e) implies (c), completing the proof. ∎

The equivalence between (a) and (c) in the previous result shows that every unitary operator is an isometry.

Next we have a result giving conditions that are equivalent to being a unitary operator. Notice that parts (d) and (e) of the previous result refer to *orthonormal families*, but parts (f) and (g) of the following result refer to *orthonormal bases*.

10.61 *unitary operators and their adjoints are isometries*

Suppose T is a bounded operator on a Hilbert space V. Then the following are equivalent:

(a) T is unitary.

(b) T is a surjective isometry.

(c) T and T^* are both isometries.

(d) T^* is unitary.

(e) T is invertible and $T^{-1} = T^*$.

(f) $\{Te_k\}_{k\in\Gamma}$ is an orthonormal basis of V for every orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V.

(g) $\{Te_k\}_{k\in\Gamma}$ is an orthonormal basis of V for some orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V.

Proof The equivalence of (a), (d), and (e) follows easily from the definition of unitary.

The equivalence of (a) and (c) follows from the equivalence in 10.60 of (a) and (c).

To prove that (a) implies (b), suppose (a) holds, so T is unitary. As we have already noted, this implies that T is an isometry. Also, the equation $TT^* = I$ implies that T is surjective. Thus (b) holds, proving that (a) implies (b).

Now suppose (b) holds, so T is a surjective isometry. Because T is surjective and injective, T is invertible. The equation $T^*T = I$ [which follows from the equivalence in 10.60 of (a) and (c)] now implies that $T^{-1} = T^*$. Thus (b) implies (e). Hence at this stage of the proof, we know that (a), (b), (c), (d), and (e) are all equivalent to each other.

To prove that (b) implies (f), suppose (b) holds, so T is a surjective isometry. Suppose $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of V. The equivalence in 10.60 of (a) and (d) implies that $\{Te_k\}_{k\in\Gamma}$ is an orthonormal family. Because $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis of V and T is surjective, the closure of the span of $\{Te_k\}_{k\in\Gamma}$ equals V. Thus $\{Te_k\}_{k\in\Gamma}$ is an orthonormal basis of V, which proves that (b) implies (f).

Obviously (f) implies (g).

Now suppose (g) holds. The equivalence in 10.60 of (a) and (e) implies that T is an isometry, which implies that the range of T is closed. Because $\{Te_k\}_{k\in\Gamma}$ is an orthonormal basis of V, the closure of the range of T equals V. Thus T is a surjective isometry, proving that (g) implies (b) and completing the proof that (a) through (g) are all equivalent to each other. ∎

The equations $T^*T = TT^* = I$ are analogous to the equation $|z|^2 = 1$ for $z \in \mathbf{C}$. We now extend this analogy to the behavior of the spectrum of a unitary operator.

10.62 *spectrum of a unitary operator*

Suppose T is a unitary operator on a Hilbert space. Then

$$\operatorname{sp}(T) \subset \{\alpha \in \mathbf{F} : |\alpha| = 1\}.$$

Proof Suppose $\alpha \in \mathbf{F}$ with $|\alpha| \neq 1$. Then

$$\begin{aligned}(T - \alpha I)^*(T - \alpha I) &= (T^* - \overline{\alpha} I)(T - \alpha I) \\ &= (1 + |\alpha|^2)I - (\alpha T^* + \overline{\alpha} T) \\ &= (1 + |\alpha|^2)\Big(I - \frac{\alpha T^* + \overline{\alpha} T}{1 + |\alpha|^2}\Big). \qquad \text{10.63}\end{aligned}$$

Looking at the last term in parentheses above, we have

$$\text{10.64} \qquad \Big\| \frac{\alpha T^* + \overline{\alpha} T}{1 + |\alpha|^2} \Big\| \leq \frac{2|\alpha|}{1 + |\alpha|^2} < 1,$$

where the last inequality holds because $|\alpha| \neq 1$. Now 10.64, 10.63, and 10.22 imply that $(T - \alpha I)^*(T - \alpha I)$ is invertible. Thus $T - \alpha I$ is left invertible. Because $T - \alpha I$ is normal, this implies that $T - \alpha I$ is invertible (see 10.55). Hence $\alpha \notin \operatorname{sp}(T)$. Thus $\operatorname{sp}(T) \subset \{\alpha \in \mathbf{F} : |\alpha| = 1\}$, as desired. ■

As a special case of the next result, we can conclude (without doing any calculations!) that the spectrum of the right shift on ℓ^2 is $\{\alpha \in \mathbf{F} : |\alpha| \leq 1\}$.

10.65 *spectrum of an isometry*

Suppose T is an isometry on a Hilbert space and T is not unitary. Then

$$\operatorname{sp}(T) = \{\alpha \in \mathbf{F} : |\alpha| \leq 1\}.$$

Proof Because T is an isometry but is not unitary, we know that T is not surjective [by the equivalence of (a) and (b) in 10.61]. In particular, T is not invertible. Thus T^* is not invertible.

Suppose $\alpha \in \mathbf{F}$ with $|\alpha| < 1$. Because $T^*T = I$, we have

$$T^*(T - \alpha I) = I - \alpha T^*.$$

The right side of the equation above is invertible (by 10.22). If $T - \alpha I$ were invertible, then the equation above would imply $T^* = (I - \alpha T^*)(T - \alpha I)^{-1}$, which would make T^* invertible as the product of invertible operators. However, the paragraph above shows T^* is not invertible. Thus $T - \alpha I$ is not invertible. Hence $\alpha \in \operatorname{sp}(T)$.

Thus $\{\alpha \in \mathbf{F} : |\alpha| < 1\} \subset \operatorname{sp}(T)$. Because $\operatorname{sp}(T)$ is closed (see 10.36), this implies $\{\alpha \in \mathbf{F} : |\alpha| \leq 1\} \subset \operatorname{sp}(T)$. The inclusion in the other direction follows from 10.34(a). Thus $\operatorname{sp}(T) = \{\alpha \in \mathbf{F} : |\alpha| \leq 1\}$. ■

EXERCISES 10B

1 Verify all the assertions in Example 10.33.

2 Suppose T is a bounded operator on a Hilbert space V.

(a) Prove that $\operatorname{sp}(S^{-1}TS) = \operatorname{sp}(T)$ for all bounded invertible operators S on V.

(b) Prove that $\operatorname{sp}(T^*) = \{\overline{\alpha} : \alpha \in \operatorname{sp}(T)\}$.

(c) Prove that if T is invertible, then $\operatorname{sp}(T^{-1}) = \left\{\frac{1}{\alpha} : \alpha \in \operatorname{sp}(T)\right\}$.

3 Suppose E is a bounded subset of $\mathbf{F}$. Show that there exists a Hilbert space V and $T \in \mathcal{B}(V)$ such that the set of eigenvalues of T equals E.

4 Suppose E is a nonempty closed bounded subset of $\mathbf{F}$. Show that there exists $T \in \mathcal{B}(\ell^2)$ such that $\operatorname{sp}(T) = E$.

5 Give an example of a bounded operator T on a normed vector space such that for every $\alpha \in \mathbf{F}$, the operator $T - \alpha I$ is not invertible.

6 Suppose T is a bounded operator on a complex nonzero Banach space V.

(a) Prove that the function

$$\alpha \mapsto \varphi\big((T - \alpha I)^{-1} f\big)$$

is analytic on $\mathbf{C} \setminus \operatorname{sp}(T)$ for every $f \in V$ and every $\varphi \in V'$.

(b) Prove that $\operatorname{sp}(T) \neq \varnothing$.

7 Prove that if T is an operator on a Hilbert space V such that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in V$, then T is a bounded operator.

8 Suppose P is a bounded operator on a Hilbert space V such that $P^2 = P$. Prove that P is self-adjoint if and only if there exists a closed subspace U of V such that $P = P_U$.

9 Suppose V is a real Hilbert space and $T \in \mathcal{B}(V)$. The *complexification* of T is the function $T_{\mathbf{C}} \colon V_{\mathbf{C}} \to V_{\mathbf{C}}$ defined by

$$T_{\mathbf{C}}(f + ig) = Tf + iTg$$

for $f, g \in V$ (see Exercise 4 in Section 8B for the definition of $V_{\mathbf{C}}$).

(a) Show that $T_{\mathbf{C}}$ is a bounded operator on the complex Hilbert space $V_{\mathbf{C}}$ and $\|T_{\mathbf{C}}\| = \|T\|$.

(b) Show that $T_{\mathbf{C}}$ is invertible if and only if T is invertible.

(c) Show that $(T_{\mathbf{C}})^* = (T^*)_{\mathbf{C}}$.

(d) Show that T is self-adjoint if and only if $T_{\mathbf{C}}$ is self-adjoint.

(e) Use the previous parts of this exercise and 10.49 and 10.38 to show that if T is self-adjoint and $V \neq \{0\}$, then $\operatorname{sp}(T) \neq \varnothing$.

10 Suppose T is a bounded operator on a Hilbert space V such that $\langle Tf, f\rangle \geq 0$ for all $f \in V$. Prove that $\operatorname{sp}(T) \subset [0,\infty)$.

11 Suppose P is a bounded operator on a Hilbert space V such that $P^2 = P$. Prove that P is self-adjoint if and only if P is normal.

12 Prove that a normal operator on a separable Hilbert space has at most countably many eigenvalues.

13 Prove or give a counterexample: If T is a normal operator on a Hilbert space and $T = A + iB$, where A and B are self-adjoint, then $\|T\| = \sqrt{\|A\|^2 + \|B\|^2}$.

A number $\alpha \in \mathbf{F}$ is called an* approximate eigenvalue *of a bounded operator T on a Hilbert space V if

$$\inf\{\|(T - \alpha I)f\| : f \in V \text{ and } \|f\| = 1\} = 0.$$

14 Suppose T is a normal operator on a Hilbert space and $\alpha \in \mathbf{F}$. Prove that $\alpha \in \operatorname{sp}(T)$ if and only if α is an approximate eigenvalue of T.

15 Suppose T is a normal operator on a Hilbert space.

(a) Prove that if α is an eigenvalue of T, then $|\alpha|^2$ is an eigenvalue of T^*T.

(b) Prove that if $\alpha \in \operatorname{sp}(T)$, then $|\alpha|^2 \in \operatorname{sp}(T^*T)$.

16 Suppose $\{e_k\}_{k\in\mathbf{Z}^+}$ is an orthonormal basis of a Hilbert space V. Suppose also that T is a normal operator on V and e_k is an eigenvector of T for every $k \geq 2$. Prove that e_1 is an eigenvector of T.

17 Prove that if T is a self-adjoint operator on a Hilbert space, then $\|T^n\| = \|T\|^n$ for every $n \in \mathbf{Z}^+$.

18 Prove that if T is a normal operator on a Hilbert space, then $\|T^n\| = \|T\|^n$ for every $n \in \mathbf{Z}^+$.

19 Suppose T is an invertible operator on a Hilbert space. Prove that T is unitary if and only if $\|T\| = \|T^{-1}\| = 1$.

20 Suppose T is a bounded operator on a complex Hilbert space, with $T = A + iB$, where A and B are self-adjoint (see 10.54). Prove that T is unitary if and only if T is normal and $A^2 + B^2 = I$.
[*If $z = x + yi$, where $x, y \in \mathbf{R}$, then $|z| = 1$ if and only if $x^2 + y^2 = 1$. Thus this exercise strengthens the analogy between the unit circle in the complex plane and the unitary operators.*]

21 Suppose T is a unitary operator on a complex Hilbert space such that $T - I$ is invertible. Prove that

$$i(T + I)(T - I)^{-1}$$

is a self-adjoint operator.
[*The function $z \mapsto i(z+1)(z-1)^{-1}$ maps $\{z \in \mathbf{C} : |z| = 1\} \setminus \{1\}$ to $\mathbf{R}$. Thus this exercise provides another useful illustration of the analogies showing unitary $\approx \{z \in \mathbf{C} : |z| = 1\}$ and self-adjoint $\approx \mathbf{R}$.*]

22 Suppose T is a self-adjoint operator on a complex Hilbert space. Prove that

$$(T + iI)(T - iI)^{-1}$$

is a unitary operator.
[*The function* $z \mapsto (z+i)(z-i)^{-1}$ *maps* $\mathbf{R}$ *to* $\{z \in \mathbf{C} : |z| = 1\} \setminus \{1\}$. *Thus this exercise provides another useful illustration of the analogies showing* (*a*) *unitary* $\iff \{z \in \mathbf{C} : |z| = 1\}$; (*b*) *self-adjoint* $\iff \mathbf{R}$.]

For T a bounded operator on a Banach space, define e^T by

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}.$$

23 (a) Prove that if T is a bounded operator on a Banach space V, then the infinite sum above converges in $\mathcal{B}(V)$ and $\|e^T\| \leq e^{\|T\|}$.

(b) Prove that if S, T are bounded operators on a Banach space V such that $ST = TS$, then $e^S e^T = e^{S+T}$.

(c) Prove that if T is a self-adjoint operator on a complex Hilbert space, then e^{iT} is unitary.

A bounded operator T on a Hilbert space is called a* partial isometry *if

$$\|Tf\| = \|f\| \textit{ for all } f \in (\operatorname{null} T)^{\perp}.$$

24 Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $h \in L^\infty(\mu)$. As usual, let $M_h \in \mathcal{B}\big(L^2(\mu)\big)$ denote the multiplication operator defined by $M_h f = fh$. Prove that M_h is a partial isometry if and only if there exists a set $E \in \mathcal{S}$ such that $h = \chi_E$.

25 Suppose T is an isometry on a Hilbert space. Prove that T^* is a partial isometry.

26 Suppose T is a bounded operator on a Hilbert space V. Prove that T is a partial isometry if and only if $T^*T = P_U$ for some closed subspace U of V.

10C Compact Operators

The Ideal of Compact Operators

A rich theory describes the behavior of compact operators, which we now define.

10.66 Definition ***compact operator***

- An operator T on a Hilbert space V is called *compact* if for every bounded sequence $f_1, f_2, \ldots$ in V, the sequence $Tf_1, Tf_2, \ldots$ has a convergent subsequence.
- The collection of compact operators on V is denoted by $\mathcal{C}(V)$.

The next result provides a large class of examples of compact operators. We will see more examples after proving a few more results.

10.67 ***bounded operators with finite-dimensional range are compact***

If T is a bounded operator on a Hilbert space and range T is finite-dimensional, then T is compact.

Proof Suppose T is a bounded operator on a Hilbert space V and range T is finite-dimensional. Suppose $e_1, \ldots, e_m$ is an orthonormal basis of range T (a finite orthonormal basis of range T exists because the Gram–Schmidt process applied to any basis of range T produces an orthonormal basis; see the proof of 8.67).

Now suppose $f_1, f_2, \ldots$ is a bounded sequence in V. For each $n \in \mathbf{Z}^+$, we have

$$Tf_n = \langle Tf_n, e_1 \rangle e_1 + \cdots + \langle Tf_n, e_m \rangle e_m.$$

The Cauchy–Schwarz inequality shows that $|\langle Tf_n, e_j \rangle| \leq \|T\| \sup_{k \in \mathbf{Z}^+} \|f_k\|$ for every $n \in \mathbf{Z}^+$ and $j \in \{1, \ldots, m\}$. Thus there exists a subsequence $f_{n_1}, f_{n_2}, \ldots$ such that $\lim_{k \to \infty} \langle Tf_{n_k}, e_j \rangle$ exists in $\mathbf{F}$ for each $j \in \{1, \ldots, m\}$. The equation displayed above now implies that $\lim_{k \to \infty} Tf_{n_k}$ exists in V. Thus T is compact. ∎

Not every bounded operator is compact. For example, the identity map on an infinite-dimensional Hilbert space is not compact (to see this, consider an orthonormal sequence, which does not have a convergent subsequence because the distance between any two distinct elements of the orthonormal sequence is $\sqrt{2}$).

10.68 ***compact operators are bounded***

Every compact operator on a Hilbert space is a bounded operator.

Proof We show that if T is an operator that is not bounded, then T is not compact. To do this, suppose V is a Hilbert space and T is an operator on V that is not bounded. Thus there exists a bounded sequence $f_1, f_2, \ldots$ in V such that $\lim_{n \to \infty} \|Tf_n\| = \infty$. Hence no subsequence of $Tf_1, Tf_2, \ldots$ converges, which means T is not compact. ∎

If V is a Hilbert space, then a *two-sided ideal* of $\mathcal{B}(V)$ is a subspace of $\mathcal{B}(V)$ that is closed under multiplication on either side by bounded operators on V. The next result states that the set of compact operators on V is a two-sided ideal of $\mathcal{B}(V)$ that is closed in the topology on $\mathcal{B}(V)$ that comes from the norm.

If V is finite-dimensional, then the only two-sided ideals of $\mathcal{B}(V)$ are $\{0\}$ and $\mathcal{B}(V)$. In contrast, if V is infinite-dimensional, then the next result shows that $\mathcal{B}(V)$ has a closed two-sided ideal that is neither $\{0\}$ nor V.

10.69 $\mathcal{C}(V)$ *is a closed two-sided ideal of* $\mathcal{B}(V)$

Suppose V is a Hilbert space.

(a) $\mathcal{C}(V)$ is a closed subspace of $\mathcal{B}(V)$.

(b) If $T \in \mathcal{C}(V)$ and $S \in \mathcal{B}(V)$, then $ST \in \mathcal{C}(V)$ and $TS \in \mathcal{C}(V)$.

Proof Suppose $f_1, f_2, \dots$ is a bounded sequence in V.

To prove that $\mathcal{C}(V)$ is closed under addition, suppose $S, T \in \mathcal{C}(V)$. Because S is compact, $Sf_1, Sf_2, \dots$ has a convergent subsequence $Sf_{n_1}, Sf_{n_2}, \dots$. Because T is compact, some subsequence of $Tf_{n_1}, Tf_{n_2}, \dots$ converges. Thus we have a subsequence of $(S+T)f_1, (S+T)f_2, \dots$ that converges. Hence $S + T \in \mathcal{C}(V)$.

The proof that $\mathcal{C}(V)$ is closed under scalar multiplication is easier and is left to the reader. Thus we now know that $\mathcal{C}(V)$ is a subspace of $\mathcal{B}(V)$.

To show that $\mathcal{C}(V)$ is closed in $\mathcal{B}(V)$, suppose $T \in \mathcal{B}(V)$ and there is a sequence $T_1, T_2, \dots$ in $\mathcal{C}(V)$ such that $\lim_{m\to\infty}\|T - T_m\| = 0$. To show that T is compact, we need to show that $Tf_{n_1}, Tf_{n_2}, \dots$ is a Cauchy sequence for some increasing sequence of positive integers $n_1 < n_2 < \cdots$.

Because T_1 is compact, there is an infinite set $Z_1 \subset \mathbf{Z}^+$ with $\|T_1 f_j - T_1 f_k\| < 1$ for all $j, k \in Z_1$. Let n_1 be the smallest element of Z_1.

Now suppose $m \in \mathbf{Z}^+$ with $m > 1$ and an infinite set $Z_{m-1} \subset \mathbf{Z}^+$ and $n_{m-1} \in Z_{m-1}$ have been chosen. Because T_m is compact, there is an infinite set $Z_m \subset Z_{m-1}$ with

$$\|T_m f_j - T_m f_k\| < \tfrac{1}{m}$$

for all $j, k \in Z_m$. Let n_m be the smallest element of Z_m such that $n_m > n_{m-1}$.

Thus we produce an increasing sequence $n_1 < n_2 < \cdots$ of positive integers and a decreasing sequence $Z_1 \supset Z_2 \supset \cdots$ of infinite subsets of $\mathbf{Z}^+$.

If $m \in \mathbf{Z}^+$ and $j, k \geq m$, then

$$\begin{aligned}\|Tf_{n_j} - Tf_{n_k}\| &\leq \|Tf_{n_j} - T_m f_{n_j}\| + \|T_m f_{n_j} - T_m f_{n_k}\| + \|T_m f_{n_k} - Tf_{n_k}\| \\ &\leq \|T - T_m\|(\|f_{n_j}\| + \|f_{n_k}\|) + \tfrac{1}{m}.\end{aligned}$$

We can make the first term on the last line above as small as we want by choosing m large (because $\lim_{m\to\infty}\|T - T_m\| = 0$ and the sequence $f_1, f_2, \dots$ is bounded). Thus $Tf_{n_1}, Tf_{n_2}, \dots$ is a Cauchy sequence, as desired, completing the proof of (a).

To prove (b), suppose $T \in \mathcal{C}(V)$ and $S \in \mathcal{B}(V)$. Hence some subsequence of $Tf_1, Tf_2, \dots$ converges, and applying S to that subsequence gives another convergent sequence. Thus $ST \in \mathcal{C}(V)$. Similarly, $Sf_1, Sf_2, \dots$ is a bounded sequence, and thus $T(Sf_1), T(Sf_2), \dots$ has a convergent subsequence; thus $TS \in \mathcal{C}(V)$. ■

The previous result now allows us to see many new examples of compact operators.

10.70 ***compact integral operators***

Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space, $K \in \mathcal{L}^2(\mu \times \mu)$, and $\mathcal{I}_K$ is the integral operator on $L^2(\mu)$ defined by

$$(\mathcal{I}_K f)(x) = \int_X K(x, y) f(y) \, d\mu(y)$$

for $f \in L^2(\mu)$ and $x \in X$. Then $\mathcal{I}_K$ is a compact operator.

Proof Example 10.5 shows that $\mathcal{I}_K$ is a bounded operator on $L^2(\mu)$.

First consider the case where there exist $g, h \in L^2(\mu)$ such that

10.71 $$K(x, y) = g(x)h(y)$$

for almost every $(x, y) \in X \times X$. In that case, if $f \in L^2(\mu)$ then

$$(\mathcal{I}_K f)(x) = \int_X g(x)h(y)f(y) \, d\mu(y) = \langle f, \overline{h} \rangle g(x)$$

for almost every $x \in X$. Thus $\mathcal{I}_K f = \langle f, \overline{h} \rangle g$. In other words, $\mathcal{I}_K$ has a one-dimensional range in this case (or a zero-dimensional range if $g = 0$). Hence 10.67 implies that $\mathcal{I}_K$ is compact.

Now consider the case where K is a finite sum of functions of the form given by the right side of 10.71. Then because the set of compact operators on V is closed under addition [by 10.69(a)], the operator $\mathcal{I}_K$ is compact in this case.

Next, consider the case of $K \in L^2(\mu \times \mu)$ such that K is the limit in $L^2(\mu \times \mu)$ of a sequence of functions $K_1, K_2, \ldots$, each of which is of the form discussed in the previous paragraph. Then

$$\|\mathcal{I}_K - \mathcal{I}_{K_n}\| = \|\mathcal{I}_{K - K_n}\| \leq \|K - K_n\|_2,$$

where the inequality above comes from 10.8. Thus $\mathcal{I}_K = \lim_{n \to \infty} \mathcal{I}_{K_n}$. By the previous paragraph, each $\mathcal{I}_{K_n}$ is compact. Because the set of compact operators is a closed subset of $\mathcal{B}(V)$ [by 10.69(a)], we conclude that $\mathcal{I}_K$ is compact.

We finish the proof by showing that the case considered in the previous paragraph includes all $K \in L^2(\mu \times \mu)$. To do this, suppose $F \in L^2(\mu \times \mu)$ is orthogonal to all the elements of $L^2(\mu \times \mu)$ of the form considered in the previous paragraph. Thus

$$0 = \int_{X \times X} g(x)h(y)\overline{F(x, y)} \, d(\mu \times \mu)(x, y) = \int_X g(x) \int_X h(y)\overline{F(x, y)} \, d\mu(y) \, d\mu(x)$$

for all $g, h \in L^2(\mu)$ where we have used Tonelli's Theorem, Fubini's Theorem, and Hölder's inequality (with $p = 2$). For fixed $h \in L^2(\mu)$, the right side above equalling 0 for all $g \in L^2(\mu)$ implies that

$$\int_X h(y)\overline{F(x, y)} \, d\mu(y) = 0$$

for almost every $x \in X$. Now $F(x, y) = 0$ for almost every $(x, y) \in X \times X$ [because the equation above holds for all $h \in L^2(\mu)$], which by 8.42 completes the proof. ■

As a special case of the previous result, we can now see that the Volterra operator $\mathcal{V}\colon L^2([0,1]) \to L^2([0,1])$ defined by

$$(\mathcal{V}f)(x) = \int_0^x f$$

is compact. This holds because, as shown in Example 10.15, the Volterra operator is an integral operator of the type considered in the previous result.

The Volterra operator is injective [because differentiating both sides of the equation $\int_0^x f = 0$ with respect to x and using the Lebesgue Differentiation Theorem (4.19) shows that $f = 0$]. Thus the Volterra operator is an example of a compact operator with infinite-dimensional range. The next example provides another class of compact operators that do not necessarily have finite-dimensional range.

10.72 Example *compact multiplication operators on ℓ^2*

Suppose $b_1, b_2, \dots$ is a sequence in $\mathbf{F}$ such that $\lim_{n\to\infty} b_n = 0$. Define a bounded linear map $T\colon \ell^2 \to \ell^2$ by

$$T(a_1, a_2, \dots) = (a_1 b_1, a_2 b_2, \dots)$$

and for $n \in \mathbf{Z}^+$, define a bounded linear map $T_n\colon \ell^2 \to \ell^2$ by

$$T_n(a_1, a_2, \dots) = (a_1 b_1, a_2 b_2, \dots, a_n b_n, 0, 0, \dots).$$

Note that each T_n is a bounded operator with finite-dimensional range and thus is compact (by 10.67). The condition $\lim_{n\to\infty} b_n = 0$ implies that $\lim_{n\to\infty} T_n = T$. Thus T is compact because $\mathcal{C}(V)$ is a closed subset of $\mathcal{B}(V)$ [by 10.69(a)].

The next result states that an operator is compact if and only if its adjoint is compact.

10.73 *T compact $\iff$ T^* compact*

Suppose T is a bounded operator on a Hilbert space. Then T is compact if and only if T^* is compact.

Proof First suppose T is compact. We want to prove that T^* is compact. To do this, suppose $f_1, f_2, \dots$ is a bounded sequence in V. Because TT^* is compact [by 10.69(b)], some subsequence $TT^* f_{n_1}, TT^* f_{n_2}, \dots$ converges. Now

$$\begin{aligned}
\|T^* f_{n_j} - T^* f_{n_k}\|^2 &= \langle T^*(f_{n_j} - f_{n_k}), T^*(f_{n_j} - f_{n_k}) \rangle \\
&= \langle TT^*(f_{n_j} - f_{n_k}), f_{n_j} - f_{n_k} \rangle \\
&\le \|TT^*(f_{n_j} - f_{n_k})\| \, \|f_{n_j} - f_{n_k}\|.
\end{aligned}$$

The inequality above implies that $T^* f_{n_1}, T^* f_{n_2}, \dots$ is a Cauchy sequence and hence converges. Thus T^* is a compact operator, completing the proof that if T is compact, then T^* is compact.

Now suppose T^* is compact. By the result proved in the paragraph above, $(T^*)^*$ is compact. Because $(T^*)^* = T$ (see 10.11), we conclude that T is compact. ■

Spectrum of Compact Operator and Fredholm Alternative

We noted earlier that the identity map on an infinite-dimensional Hilbert space is not compact. The next result shows that much more is true.

10.74 ***no infinite-dimensional closed subspace in range of compact operator***

The range of each compact operator on a Hilbert space contains no infinite-dimensional closed subspaces.

Proof Suppose T is a bounded operator on a Hilbert space V and U is an infinite-dimensional closed subspace contained in range T. We want to show that T is not compact.

Because T is a continuous operator, $T^{-1}(U)$ is a closed subspace of V. Let $S = T|_{T^{-1}(U)}$. Thus S is a surjective bounded linear map from the Hilbert space $T^{-1}(U)$ onto the Hilbert space U [here $T^{-1}(U)$ and U are Hilbert spaces by 6.16(b)]. The Open Mapping Theorem (6.81) implies S maps the open unit ball of $T^{-1}(U)$ to an open subset of U. Thus there exists $r > 0$ such that

10.75 $$\{g \in U : \|g\| < r\} \subset \{Tf : f \in T^{-1}(U) \text{ and } \|f\| < 1\}.$$

Because U is an infinite-dimensional Hilbert space, there exists an orthonormal sequence $e_1, e_2, \dots$ in U, as can be seen by applying the Gram–Schmidt process (see the proof of 8.67) to any linearly independent sequence in U. Each $\frac{re_n}{2}$ is in the left side of 10.75. Thus for each $n \in \mathbf{Z}^+$, there exists $f_n \in T^{-1}(U)$ such that $\|f_n\| < 1$ and $Tf_n = \frac{re_n}{2}$. The sequence $f_1, f_2, \dots$ is bounded, but the sequence $Tf_1, Tf_2, \dots$ has no convergent subsequence because $\left\|\frac{re_j}{2} - \frac{re_k}{2}\right\| = \frac{\sqrt{2}r}{2}$ for $j \neq k$. Thus T is not compact, as desired. ■

Suppose T is a compact operator on an infinite-dimensional Hilbert space. The result above implies that T is not surjective. In particular, T is not invertible. Thus we have the following result.

10.76 ***compact implies not invertible on infinite-dimensional Hilbert spaces***

If T is a compact operator on an infinite-dimensional Hilbert space, then $0 \in \operatorname{sp}(T)$.

Although 10.74 shows that if T is compact then range T contains no infinite-dimensional closed subspaces, the next result shows that the situation differs drastically for $T - \alpha I$ if $\alpha \in \mathbf{F} \setminus \{0\}$.

The proof of the next result makes use of the restriction of $T - \alpha I$ to the closed subspace $\big(\operatorname{null}(T - \alpha I)\big)^{\perp}$. As motivation for considering this restriction, recall that each $f \in V$ can be written uniquely as $f = g + h$, where $g \in \operatorname{null}(T - \alpha I)$ and $h \in \big(\operatorname{null}(T - \alpha I)\big)^{\perp}$ (see 8.43). Thus $(T - \alpha I)f = (T - \alpha I)h$, which implies that $\operatorname{range}(T - \alpha I) = (T - \alpha I)\Big(\big(\operatorname{null}(T - \alpha I)\big)^{\perp}\Big)$.

10.77 *closed range*

If T is a compact operator on a Hilbert space, then $T - \alpha I$ has closed range for every $\alpha \in \mathbf{F}$ with $\alpha \neq 0$.

Proof Suppose T is a compact operator on a Hilbert space V and $\alpha \in \mathbf{F}$ is such that $\alpha \neq 0$.

10.78 Claim: there exists $r > 0$ such that

$$\|f\| \leq r\|(T - \alpha I)f\| \text{ for all } f \in \big(\operatorname{null}(T - \alpha I)\big)^{\perp}.$$

To prove the claim above, suppose it is false. Then for each $n \in \mathbf{Z}^+$, there exists $f_n \in \big(\operatorname{null}(T - \alpha I)\big)^{\perp}$ such that

$$\|f_n\| = 1 \quad \text{and} \quad \|(T - \alpha I)f_n\| < \tfrac{1}{n}.$$

Because T is compact, there exists a subsequence $Tf_{n_1}, Tf_{n_2}, \ldots$ such that

10.79
$$\lim_{k \to \infty} Tf_{n_k} = g$$

for some $g \in V$. Subtracting the equation

10.80
$$\lim_{k \to \infty} (T - \alpha I)f_{n_k} = 0$$

from 10.79 and then dividing by α shows that

$$\lim_{k \to \infty} f_{n_k} = \tfrac{1}{\alpha}g.$$

The equation above implies $\|g\| = |\alpha|$; hence $g \neq 0$. Each $f_{n_k} \in \big(\operatorname{null}(T - \alpha I)\big)^{\perp}$; hence we also conclude that $g \in \big(\operatorname{null}(T - \alpha I)\big)^{\perp}$. Applying $T - \alpha I$ to both sides of the equation above and using 10.80 shows that $g \in \operatorname{null}(T - \alpha I)$. Thus g is a nonzero element of both $\operatorname{null}(T - \alpha I)$ and its orthogonal complement. This contradiction completes the proof of the claim in 10.78.

To show that $\operatorname{range}(T - \alpha I)$ is closed, suppose $h_1, h_2, \ldots$ is a sequence in $\operatorname{range}(T - \alpha I)$ that converges to some $h \in V$. For each $n \in \mathbf{Z}^+$, there exists $f_n \in \big(\operatorname{null}(T - \alpha I)\big)^{\perp}$ such that $(T - \alpha I)f_n = h_n$. Because $h_1, h_2, \ldots$ is a Cauchy sequence, 10.78 shows that $f_1, f_2, \ldots$ is also a Cauchy sequence. Thus there exists $f \in V$ such that $\lim_{n \to \infty} f_n = f$, which implies $h = (T - \alpha I)f \in \operatorname{range}(T - \alpha I)$. Hence $\operatorname{range}(T - \alpha I)$ is closed. ■

Suppose T is a compact operator on a Hilbert space V and $f \in V$ and $\alpha \in \mathbf{F} \setminus \{0\}$. An immediate consequence (often useful when investigating integral equations) of the result above and 10.13(d) is that the equation

$$Tg - \alpha g = f$$

has a solution $g \in V$ if and only if $\langle f, h \rangle = 0$ for every $h \in V$ such that $T^*h = \overline{\alpha}h$.

10.81 Definition ***geometric multiplicity***

- The *geometric multiplicity* of an eigenvalue α of an operator T is defined to be the dimension of $\operatorname{null}(T - \alpha I)$.
- In other words, the geometric multiplicity of an eigenvalue α of T is the dimension of the subspace consisting of 0 and all the eigenvectors of T corresponding to α.

There exist compact operators for which the eigenvalue 0 has infinite geometric multiplicity. The next result shows that this cannot happen for nonzero eigenvalues.

10.82 ***nonzero eigenvalues of compact operators have finite multiplicity***

Suppose T is a compact operator on a Hilbert space and $\alpha \in \mathbf{F}$ with $\alpha \neq 0$. Then $\operatorname{null}(T - \alpha I)$ is finite-dimensional.

Proof Suppose $f \in \operatorname{null}(T - \alpha I)$. Then $f = T\left(\frac{f}{\alpha}\right)$. Hence $f \in \operatorname{range} T$.

Thus we have shown that $\operatorname{null}(T - \alpha I) \subset \operatorname{range} T$. Because T is continuous, $\operatorname{null}(T - \alpha I)$ is closed. Thus 10.74 implies that $\operatorname{null}(T - \alpha I)$ is finite-dimensional. ■

The next lemma is used in our proof of the Fredholm Alternative (10.85). Note that this lemma implies that every injective operator on a finite-dimensional vector space is surjective (because a finite-dimensional vector space cannot have an infinite chain of strictly decreasing subspaces—the dimension decreases by at least 1 in each step). Also, see Exercise 10 for the analogous result implying that every surjective operator on a finite-dimensional vector space is injective.

10.83 ***injective but not surjective***

If T is an injective but not surjective operator on a vector space, then

$$\operatorname{range} T \supsetneq \operatorname{range} T^2 \supsetneq \operatorname{range} T^3 \supsetneq \cdots .$$

Proof Suppose T is an injective but not surjective operator on a vector space V. Suppose $n \in \mathbf{Z}^+$. If $g \in V$, then

$$T^{n+1}g = T^n(Tg) \in \operatorname{range} T^n.$$

Thus $\operatorname{range} T^n \supset \operatorname{range} T^{n+1}$.

To show that the last inclusion is not an equality, note that because T is not surjective, there exists $f \in V$ such that

10.84 $$f \notin \operatorname{range} T.$$

Now $T^n f \in \operatorname{range} T^n$. However, $T^n f \notin \operatorname{range} T^{n+1}$ because if $g \in V$ and $T^n f = T^{n+1}g$, then $T^n f = T^n(Tg)$, which would imply that $f = Tg$ (because T^n is injective), which would contradict 10.84. Thus $\operatorname{range} T^n \supsetneq \operatorname{range} T^{n+1}$. ■

Compact operators behave, in some respects, like operators on a finite-dimensional vector space. For example, the following important theorem should be familiar to you in the finite-dimensional context (where the choice of $\alpha = 0$ need not be excluded).

10.85 *Fredholm Alternative*

Suppose T is a compact operator on a Hilbert space and $\alpha \in \mathbf{F}$ with $\alpha \neq 0$. Then the following are equivalent:

(a) $\alpha \in \operatorname{sp}(T)$.

(b) α is an eigenvalue of T.

(c) $T - \alpha I$ is not surjective.

Proof Clearly (b) implies (a) and (c) implies (a).

To prove that (a) implies (b), suppose $\alpha \in \operatorname{sp}(T)$ but α is not an eigenvalue of T. Thus $T - \alpha I$ is injective but $T - \alpha I$ is not surjective. Thus 10.83 applied to $T - \alpha I$ shows that

10.86 $$\operatorname{range}(T - \alpha I) \supsetneqq \operatorname{range}(T - \alpha I)^2 \supsetneqq \operatorname{range}(T - \alpha I)^3 \supsetneqq \cdots .$$

If $n \in \mathbf{Z}^+$, then the Binomial Theorem and 10.69 show that

$$(T - \alpha I)^n = S + (-\alpha)^n I$$

for some compact operator S. Now 10.77 shows that $\operatorname{range}(T - \alpha I)^n$ is a closed subspace of the Hilbert space on which T operates. Thus 10.86 implies that for each $n \in \mathbf{Z}^+$, there exists

10.87 $$f_n \in \operatorname{range}(T - \alpha I)^n \cap \big(\operatorname{range}(T - \alpha I)^{n+1}\big)^\perp$$

such that $\|f_n\| = 1$.

Now suppose $j, k \in \mathbf{Z}^+$ with $j < k$. Then

10.88 $$Tf_j - Tf_k = (T - \alpha I)f_j - (T - \alpha I)f_k - \alpha f_k + \alpha f_j.$$

Because f_j and f_k are both in $\operatorname{range}(T - \alpha I)^j$, the first two terms on the right side of 10.88 are in $\operatorname{range}(T - \alpha I)^{j+1}$. Because $j + 1 \leq k$, the third term in 10.88 is also in $\operatorname{range}(T - \alpha I)^{j+1}$. Now 10.87 implies that the last term in 10.88 is orthogonal to the sum of the first three terms. Thus 10.88 leads to the inequality

$$\|Tf_j - Tf_k\| \geq \|\alpha f_j\| = |\alpha|.$$

The inequality above implies that $Tf_1, Tf_2, \ldots$ has no convergent subsequence, which contradicts the compactness of T. This contradiction means the assumption that α is not an eigenvalue of T was false, completing the proof that (a) implies (b).

At this stage, we know that (a) and (b) are equivalent and that (c) implies (a). To prove that (a) implies (c), suppose $\alpha \in \operatorname{sp}(T)$. Thus $\overline{\alpha} \in \operatorname{sp}(T^*)$. Applying the equivalence of (a) and (b) to T^*, we conclude that $\overline{\alpha}$ is an eigenvalue of T^*. Thus applying 10.13(d) to $T - \alpha I$ shows that $T - \alpha I$ is not surjective, completing the proof that (a) implies (c). ■

The previous result traditionally has the word *alternative* in its name because it can be rephrased as follows:

> If T is a compact operator on a Hilbert space V and $\alpha \in \mathbf{F} \setminus \{0\}$, then exactly one of the following holds:
>
> 1. the equation $Tf = \alpha f$ has a nonzero solution $f \in V$;
> 2. the equation $g = Tf - \alpha f$ has a solution $f \in V$ for every $g \in V$.

The next example shows the power of the Fredholm Alternative. In this example, we want to show that $\mathcal{V} - \alpha I$ is invertible for all $\alpha \in \mathbf{F} \setminus \{0\}$. The verification that $\mathcal{V} - \alpha I$ is injective is straightforward. Showing that $\mathcal{V} - \alpha I$ is surjective would require more work. However, the Fredholm Alternative tells us, with no further work, that $\mathcal{V} - \alpha I$ is invertible.

10.89 **Example** *spectrum of the Volterra operator*

We want to show that the spectrum of the Volterra operator $\mathcal{V}$ is $\{0\}$ (see Example 10.15 for the definition of $\mathcal{V}$). The Volterra operator $\mathcal{V}$ is compact (see the comment after the proof of 10.70). Thus $0 \in \operatorname{sp}(\mathcal{V})$, by 10.76.

Suppose $\alpha \in \mathbf{F} \setminus \{0\}$. To show that $\alpha \notin \operatorname{sp}(\mathcal{V})$, we need only show that α is not an eigenvalue of $\mathcal{V}$ (by 10.85). Thus suppose $f \in L^2([0,1])$ and $\mathcal{V}f = \alpha f$. Hence

10.90
$$\int_0^x f = \alpha f(x)$$

for almost every $x \in [0,1]$. The left side of 10.90 is a continuous function of x and thus so is the right side, which implies that f is continuous. The continuity of f now implies that the left side of 10.90 has a continuous derivative, and thus f has a continuous derivative.

Now differentiate both sides of 10.90 with respect to x, getting

$$f(x) = \alpha f'(x)$$

for all $x \in (0,1)$. Standard calculus shows that the equation above implies that

$$f(x) = ce^{x/\alpha}$$

for some constant c. However, 10.90 implies that the continuous function f must satisfy the equation $f(0) = 0$. Thus $c = 0$, which implies $f = 0$.

The conclusion of the last paragraph shows that α is not an eigenvalue of $\mathcal{V}$. The Fredholm Alternative (10.85) now shows that $\alpha \notin \operatorname{sp}(\mathcal{V})$. Thus $\operatorname{sp}(\mathcal{V}) = \{0\}$.

If α is an eigenvalue of an operator T on a finite-dimensional Hilbert space, then $\overline{\alpha}$ is an eigenvalue of T^*. This result does not hold for bounded operators on infinite-dimensional Hilbert spaces.

However, suppose T is a compact operator on a Hilbert space and α is a nonzero eigenvalue of T. Thus $\alpha \in \operatorname{sp}(T)$, which implies that $\overline{\alpha} \in \operatorname{sp}(T^*)$ (because a bounded operator is invertible if and only if its adjoint is invertible). The Fredholm Alternative (10.85) now shows that $\overline{\alpha}$ is an eigenvalue of T^*. Thus compactness allows us to recover the finite-dimensional result (except for the case $\alpha = 0$).

Our next result states that if T is a compact operator and $\alpha \neq 0$, then $\operatorname{null}(T - \alpha I)$ and $\operatorname{null}(T^* - \overline{\alpha} I)$ have the same dimension (denoted dim). This result about the dimensions of spaces of eigenvectors is easier to prove in finite dimensions. Specifically, suppose S is an operator on a finite-dimensional Hilbert space V (you can think of $S = T - \alpha I$). Then

$$\dim \operatorname{null} S = \dim V - \dim \operatorname{range} S = \dim(\operatorname{range} S)^{\perp} = \dim \operatorname{null} S^*,$$

where the justification for each step should be familiar to you from finite-dimensional linear algebra. This finite-dimensional proof does not work in infinite dimensions because the expression $\dim V - \dim \operatorname{range} S$ could be of the form $\infty - \infty$.

Although the dimensions of the two null spaces in the result below are the same, even in finite dimensions the two null spaces are not necessarily equal to each other (but we do have equality of the two null spaces when T is normal; see 10.56).

Note that both dimensions in the result below are finite (by 10.82 and 10.73).

10.91 ***null spaces of $T - \alpha I$ and $T^* - \overline{\alpha} I$ have same dimensions***

Suppose T is a compact operator on a Hilbert space and $\alpha \in \mathbf{F}$ with $\alpha \neq 0$. Then

$$\dim \operatorname{null}(T - \alpha I) = \dim \operatorname{null}(T^* - \overline{\alpha} I).$$

Proof Suppose $\dim \operatorname{null}(T - \alpha I) < \dim \operatorname{null}(T^* - \overline{\alpha} I)$. Because $\operatorname{null}(T^* - \overline{\alpha} I)$ equals $\big(\operatorname{range}(T - \alpha I)\big)^{\perp}$, there is a bounded injective linear map

$$R\colon \operatorname{null}(T - \alpha I) \to \big(\operatorname{range}(T - \alpha I)\big)^{\perp}$$

that is not surjective. Let V denote the Hilbert space on which T operates, and let P be the orthogonal projection of V onto $\operatorname{null}(T - \alpha I)$. Define a linear map $S\colon V \to V$ by

$$S = T + RP.$$

Because RP is a bounded operator with finite-dimensional range, S is compact. Also,

$$S - \alpha I = (T - \alpha I) + RP.$$

Every element of $\operatorname{range}(T - \alpha I)$ is orthogonal to every element of $\operatorname{range} RP$. Suppose $f \in V$ and $(S - \alpha I)f = 0$. The equation above shows that $(T - \alpha I)f = 0$ and $RPf = 0$. Because $f \in \operatorname{null}(T - \alpha f)$, we see that $Pf = f$, which then implies that $Rf = RPf = 0$, which then implies that $f = 0$ (because R is injective). Hence $S - \alpha I$ is injective.

However, because R maps onto a proper subset of $\big(\operatorname{range}(T - \alpha I)\big)^{\perp}$, we see that $S - \alpha I$ is not surjective, which contradicts the equivalence of (b) and (c) in 10.85. This contradiction means the assumption that $\dim \operatorname{null}(T - \alpha I) < \dim \operatorname{null}(T^* - \overline{\alpha} I)$ was false. Hence we have proved that

10.92 $$\dim \operatorname{null}(T - \alpha I) \geq \dim \operatorname{null}(T^* - \overline{\alpha} I)$$

for every compact operator T and every $\alpha \in \mathbf{F} \setminus \{0\}$.

Now apply the conclusion of the previous paragraph to T^* (which is compact by 10.73) and $\overline{\alpha}$, getting 10.92 with the inequality reversed, completing the proof. ∎

The spectrum of an operator on a finite-dimensional Hilbert space is a finite set, consisting just of the eigenvalues of the operator. The spectrum of a compact operator on an infinite-dimensional Hilbert space can be an infinite set. However, our next result implies that if a compact operator has infinite spectrum, then that spectrum consists of 0 and a sequence in $\mathbf{F}$ with limit 0.

10.93 *spectrum of a compact operator*

Suppose T is a compact operator on a Hilbert space. Then

$$\{\alpha \in \operatorname{sp}(T) : |\alpha| \geq \delta\}$$

is a finite set for every $\delta > 0$.

Proof Fix $\delta > 0$. Suppose there exist distinct $\alpha_1, \alpha_2, \ldots$ in $\operatorname{sp}(T)$ with $|\alpha_n| \geq \delta$ for every $n \in \mathbf{Z}^+$. The Fredholm Alternative (10.85) implies that each α_n is an eigenvalue of T. For $n \in \mathbf{Z}^+$, let

$$U_n = \operatorname{null}\big((T - \alpha_1 I) \cdots (T - \alpha_n I)\big).$$

and let $U_0 = \{0\}$. Because T is continuous, each U_n is a closed subspace of the Hilbert space on which T operates. Furthermore, $U_{n-1} \subset U_n$ for each $n \in \mathbf{Z}^+$ because operators of the form $T - \alpha_j I$ and $T - \alpha_k I$ commute with each other.

If $n \in \mathbf{Z}^+$ and g is an eigenvector of T corresponding to the eigenvalue α_n, then $g \in U_n$ but $g \notin U_{n-1}$ because

$$(T - \alpha_1 I) \cdots (T - \alpha_{n-1} I)g = (\alpha_n - \alpha_1) \cdots (\alpha_n - \alpha_{n-1})g \neq 0.$$

In other words, we have

$$U_1 \subsetneqq U_2 \subsetneqq U_3 \subsetneqq \cdots .$$

Thus for each $n \in \mathbf{Z}^+$, there exists

10.94 $$e_n \in U_n \cap (U_{n-1}{}^{\perp})$$

such that $\|e_n\| = 1$.

Now suppose $j, k \in \mathbf{Z}^+$ with $j < k$. Then

10.95 $$Te_j - Te_k = (T - \alpha_j I)e_j - (T - \alpha_k I)e_k + \alpha_j e_j - \alpha_k e_k.$$

Because $j \leq k - 1$, the first three terms on the right side of 10.95 are in U_{k-1}. Now 10.94 implies that the last term in 10.95 is orthogonal to the sum of the first three terms. Thus 10.95 leads to the inequality

$$\|Te_j - Te_k\| \geq \|\alpha_k e_k\| = |\alpha_k| \geq \delta.$$

The inequality above implies that $Te_1, Te_2, \ldots$ has no convergent subsequence, which contradicts the compactness of T. This contradiction means that the assumption that $\operatorname{sp}(T)$ contains infinitely many elements with absolute value at least δ was false. ■

EXERCISES 10C

1 Prove that if T is a compact operator on a Hilbert space V and $e_1, e_2, \ldots$ is an orthonormal sequence in V, then $\lim_{n\to\infty} Te_n = 0$.

2 Prove that if T is a compact operator on $L^2([0,1])$, then $\lim\limits_{n\to\infty} \sqrt{n}\|T(x^n)\|_2 = 0$, where x^n means the element of $L^2([0,1])$ defined by $x \mapsto x^n$.

3 Suppose T is a compact operator on a Hilbert space V and $f_1, f_2, \ldots$ is a sequence in V such that $\lim_{n\to\infty}\langle f_n, g\rangle = 0$ for every $g \in V$. Prove that $\lim_{n\to\infty}\|Tf_n\| = 0$.

4 Suppose $h \in L^\infty(\mathbf{R})$. Define $M_h \in \mathcal{B}\big(L^2(\mathbf{R})\big)$ by $M_h f = fh$. Prove that if $\|h\|_\infty > 0$, then M_h is not compact.

5 Suppose $(b_1, b_2, \ldots) \in \ell^\infty$. Define $T\colon \ell^2 \to \ell^2$ by

$$T(a_1, a_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots).$$

Prove that T is compact if and only if $\lim\limits_{n\to\infty} b_n = 0$.

6 Suppose T is a bounded operator on a Hilbert space V. Prove that if there exists an orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V such that

$$\sum_{k\in\Gamma}\|Te_k\|^2 < \infty,$$

then T is compact.

7 Suppose T is a bounded operator on a Hilbert space V. Prove that if $\{e_k\}_{k\in\Gamma}$ and $\{f_j\}_{j\in\Omega}$ are orthonormal bases of V, then

$$\sum_{k\in\Gamma}\|Te_k\|^2 = \sum_{j\in\Omega}\|Tf_j\|^2.$$

8 Suppose T is a bounded operator on a Hilbert space. Prove that T is compact if and only if T^*T is compact.

9 Prove that if T is a compact operator on an infinite-dimensional Hilbert space, then $\|I - T\| \geq 1$.

10 Show that if T is a surjective but not injective operator on a vector space V, then

$$\operatorname{null} T \subsetneqq \operatorname{null} T^2 \subsetneqq \operatorname{null} T^3 \subsetneqq \cdots .$$

11 Suppose T is a compact operator on a Hilbert space and $\alpha \in \mathbf{F}\setminus\{0\}$.

(a) Prove that $\operatorname{range}(T - \alpha I)^{m-1} = \operatorname{range}(T - \alpha I)^m$ for some $m \in \mathbf{Z}^+$.

(b) Prove that $\operatorname{null}(T - \alpha I)^{n-1} = \operatorname{null}(T - \alpha I)^n$ for some $n \in \mathbf{Z}^+$.

(c) Show that the smallest positive integer m that works in (a) equals the smallest positive integer n that works in (b).

12 Prove that if $f\colon [0,1] \to \mathbf{F}$ is a continuous function, then there exists a continuous function $g\colon [0,1] \to \mathbf{F}$ such that

$$f(x) = g(x) + \int_0^x g$$

for all $x \in [0,1]$.

13 Suppose S is a bounded invertible operator on a Hilbert space V and T is a compact operator on V.

(a) Prove that $S + T$ has closed range.

(b) Prove that $S + T$ is injective if and only if $S + T$ is surjective.

(c) Prove that $\operatorname{null}(S + T)$ and $\operatorname{null}(S^* + T^*)$ are finite-dimensional.

(d) Prove that $\dim \operatorname{null}(S + T) = \dim \operatorname{null}(S^* + T^*)$.

(e) Prove that there exists $R \in \mathcal{B}(V)$ such that range R is finite-dimensional and $S + T + R$ is invertible.

14 Suppose T is a compact operator on a Hilbert space V. Prove that range T is a separable subspace of V.

15 Suppose T is a compact operator on a Hilbert space V and $e_1, e_2, \dots$ is an orthonormal basis of $\overline{\operatorname{range} T}$. Let P_n denote the orthogonal projection of V onto $\operatorname{span}\{e_1, \dots, e_n\}$.

(a) Prove that $\lim\limits_{n\to\infty} \|T - P_n T\| = 0$.

(b) Prove that an operator on a Hilbert space V is compact if and only if it is the limit in $\mathcal{B}(V)$ of a sequence of bounded operators with finite-dimensional range.

16 Prove that if T is a compact operator on a Hilbert space V, then there exists a sequence $S_1, S_2, \dots$ of invertible operators on V such that $\lim\limits_{n\to\infty} \|T - S_n\| = 0$.

17 Suppose T is a bounded operator on a Hilbert space such that $p(T)$ is compact for some nonzero polynomial p with coefficients in $\mathbf{F}$. Prove that $\operatorname{sp}(T)$ is a countable set.

Suppose T is a bounded operator on a Hilbert space. The* algebraic multiplicity *of an eigenvalue α of T is defined to be the dimension of the subspace

$$\bigcup_{n=1}^{\infty} \operatorname{null}(T - \alpha I)^n.$$

As an easy example, if T is the left shift as defined in the next exercise, then the eigenvalue 0 of T has geometric multiplicity 1 but algebraic multiplicity ∞.

The definition above of algebraic multiplicity is equivalent on finite-dimensional spaces to the common definition involving the multiplicity of a root of the characteristic polynomial. However, the definition used here is cleaner (no determinants needed) and has the advantage of working on infinite-dimensional Hilbert spaces.

18 Suppose $T \in \mathcal{B}(\ell^2)$ is defined by $T(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$. Suppose also that $\alpha \in \mathbf{F}$ and $|\alpha| < 1$.

(a) Show that the geometric multiplicity of α as an eigenvalue of T equals 1.

(b) Show that the algebraic multiplicity of α as an eigenvalue of T equals ∞.

19 Prove that the geometric multiplicity of an eigenvalue of a normal operator on a Hilbert space equals the algebraic multiplicity of that eigenvalue.

20 Prove that every nonzero eigenvalue of a compact operator on a Hilbert space has finite algebraic multiplicity.

21 Prove that if T is a compact operator on a Hilbert space and α is a nonzero eigenvalue of T, then the algebraic multiplicity of α as an eigenvalue of T equals the algebraic multiplicity of $\overline{\alpha}$ as an eigenvalue of T^*.

22 Prove that if V is a separable Hilbert space, then the Banach space $\mathcal{C}(V)$ is separable.

10D Spectral Theorem for Compact Operators

Orthonormal Bases Consisting of Eigenvectors

We begin this section with the following useful lemma.

10.96 $T^*T - \|T\|^2 I$ ***is not invertible***

If T is a bounded operator on a nonzero Hilbert space, then $\|T\|^2 \in \operatorname{sp}(T^*T)$.

Proof Suppose T is a bounded operator on a nonzero Hilbert space V. Let $f_1, f_2, \ldots$ be a sequence in V such that $\|f_n\| = 1$ for each $n \in \mathbf{Z}^+$ and

10.97 $$\lim_{n\to\infty} \|Tf_n\| = \|T\|.$$

Then

$$\begin{aligned}\big\|T^*Tf_n - \|T\|^2 f_n\big\|^2 &= \|T^*Tf_n\|^2 - 2\|T\|^2\langle T^*Tf_n, f_n\rangle + \|T\|^4\\ &= \|T^*Tf_n\|^2 - 2\|T\|^2\,\|Tf_n\|^2 + \|T\|^4\\ &\le 2\|T\|^4 - 2\|T\|^2\,\|Tf_n\|^2, \qquad (10.98)\end{aligned}$$

where the last line holds because $\|T^*Tf_n\| \le \|T^*\|\,\|Tf_n\| \le \|T\|^2$. Now 10.97 and 10.98 imply that

$$\lim_{n\to\infty} (T^*T - \|T\|^2 I)f_n = 0.$$

Because $\|f_n\| = 1$ for each $n \in \mathbf{Z}^+$, the equation above implies that $T^*T - \|T\|^2 I$ is not invertible, as desired. ∎

The next result indicates one way in which self-adjoint compact operators behave like self-adjoint operators on finite-dimensional Hilbert spaces.

10.99 ***every self-adjoint compact operator has an eigenvalue.***

Suppose T is a self-adjoint compact operator on a nonzero Hilbert space. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

Proof Because T is self-adjoint, 10.96 states that $T^2 - \|T\|^2 I$ is not invertible. Now

$$T^2 - \|T\|^2 I = \big(T - \|T\|I\big)\big(T + \|T\|I\big).$$

Thus $T - \|T\|I$ and $T + \|T\|I$ cannot both be invertible. Hence $\|T\| \in \operatorname{sp}(T)$ or $-\|T\| \in \operatorname{sp}(T)$. Because T is compact, 10.85 now implies that $\|T\|$ or $-\|T\|$ is an eigenvalue of T, as desired, or that $\|T\| = 0$, which means that $T = 0$, in which case 0 is an eigenvalue of T. ∎

If T is an operator on a vector space V and U is a subspace of V, then $T|_U$ is a linear map from U to V. For $T|_U$ to be an operator (meaning that it is a linear map from a vector space to itself), we need $T(U) \subset U$. Thus we are led to the following definition.

10.100 Definition *invariant subspace*

Suppose T is an operator on a vector space V. A subspace U of V is called an *invariant subspace* for T if $Tf \in U$ for every $f \in U$.

10.101 Example *invariant subspaces*

You should verify each of the assertions below.

- For $b \in [0,1]$, the subspace
$$\{f \in L^2([0,1]) : f(t) = 0 \text{ for almost every } t \in [0,b]\}$$
is an invariant subspace for the Volterra operator $\mathcal{V}\colon L^2([0,1]) \to L^2([0,1])$ defined by $(\mathcal{V}f)(x) = \int_0^x f$.
- Suppose T is an operator on a Hilbert space V and $f \in V$ with $f \neq 0$. Then $\operatorname{span}\{f\}$ is an invariant subspace for T if and only if f is an eigenvector of T.
- Suppose T is an operator on a Hilbert space V. Then $\{0\}$, V, null T, and range T are invariant subspaces for T.
- If T is a bounded operator on a Hilbert space and U is an invariant subspace for T, then $\overline{U}$ is an invariant subspace for T.

If T is a compact operator on a Hilbert space and U is an invariant subspace for T, then $T|_U$ is a compact operator on U, as follows from the definitions.

If U is an invariant subspace for a self-adjoint operator T, then $T|_U$ is self-adjoint because

The most important open question in operator theory is the invariant subspace problem, which asks whether every bounded operator on a Hilbert space with dimension greater than 1 has a closed invariant subspace other than $\{0\}$ and V.

$$\langle (T|_U)f, g\rangle = \langle Tf, g\rangle = \langle f, Tg\rangle = \langle f, (T|_U)g\rangle$$

for all $f, g \in U$. The next result shows that a bit more is true.

10.102 ***U invariant for self-adjoint T implies $U^\perp$ invariant for T***

Suppose U is an invariant subspace for a self-adjoint operator T. Then

(a) $U^\perp$ is also an invariant subspace for T;

(b) $T|_{U^\perp}$ is a self-adjoint operator on $U^\perp$.

Proof To prove (a), suppose $f \in U^{\perp}$. If $g \in U$, then

$$\langle Tf, g\rangle = \langle f, Tg\rangle = 0,$$

where the first equality holds because T is self-adjoint and the second equality holds because $Tg \in U$ and $f \in U^{\perp}$. Because the equation above holds for all $g \in U$, we conclude that $Tf \in U^{\perp}$. Thus $U^{\perp}$ is an invariant subspace for T, proving (a).

By part (a), we can think of $T|_{U^{\perp}}$ as an operator on $U^{\perp}$. To prove (b), suppose $h \in U^{\perp}$. If $f \in U^{\perp}$, then

$$\langle f, (T|_{U^{\perp}})^* h\rangle = \langle T|_{U^{\perp}} f, h\rangle = \langle Tf, h\rangle = \langle f, Th\rangle = \langle f, T|_{U^{\perp}} h\rangle.$$

Because $(T|_{U^{\perp}})^* h$ and $T|_{U^{\perp}} h$ are both in $U^{\perp}$ and the equation above holds for all $f \in U^{\perp}$, we conclude that $(T|_{U^{\perp}})^* h = T|_{U^{\perp}} h$, proving (b). ■

Operators for which there exists an orthonormal basis consisting of eigenvectors may be the easiest operators to understand. The next result states that any such operator must be self-adjoint in the case of a real Hilbert space and normal in the case of a complex Hilbert space.

10.103 ***orthonormal basis of eigenvectors implies self-adjoint or normal***

Suppose T is a bounded operator on a Hilbert space V and there is an orthonormal basis of V consisting of eigenvectors of T.

(a) If $\mathbf{F} = \mathbf{R}$, then T is self-adjoint.

(b) If $\mathbf{F} = \mathbf{C}$, then T is normal.

Proof Suppose $\{e_j\}_{j\in\Gamma}$ is an orthonormal basis of V such that e_j is an eigenvector of T for each $j \in \Gamma$. Thus there exists a family $\{\alpha_j\}_{j\in\Gamma}$ in $\mathbf{F}$ such that

10.104
$$Te_j = \alpha_j e_j$$

for each $j \in \Gamma$. If $k \in \Gamma$ and $f \in V$, then

$$\begin{aligned}\langle f, T^* e_k\rangle = \langle Tf, e_k\rangle &= \Big\langle T\Big(\sum_{j\in\Gamma}\langle f, e_j\rangle e_j\Big), e_k\Big\rangle \\ &= \Big\langle \sum_{j\in\Gamma}\alpha_j\langle f, e_j\rangle e_j, e_k\Big\rangle = \alpha_k\langle f, e_k\rangle = \langle f, \overline{\alpha_k} e_k\rangle.\end{aligned}$$

The equation above implies that

10.105
$$T^* e_k = \overline{\alpha_k} e_k.$$

To prove (a), suppose $\mathbf{F} = \mathbf{R}$. Then 10.105 and 10.104 imply $T^* e_k = \alpha_k e_k = Te_k$ for each $k \in K$. Hence $T^* = T$, completing the proof of (a).

To prove (b), now suppose $\mathbf{F} = \mathbf{C}$. If $k \in \Gamma$, then 10.105 and 10.104 imply that

$$(T^*T)(e_k) = T^*(\alpha_k e_k) = |\alpha_k|^2 e_k = T(\overline{\alpha_k} e_k) = (TT^*)(e_k).$$

Because the equation above holds for all $k \in \Gamma$, we conclude that $T^*T = TT^*$. Thus T is normal, completing the proof of (b). ■

The next result is one of the major highlights of the theory of compact operators on Hilbert spaces. The result as stated below applies to both real and complex Hilbert spaces. In the case of a real Hilbert space, the result below can be combined with 10.103(a) to produce the following result: A compact operator on a real Hilbert space is self-adjoint if and only if there is an orthonormal basis of the Hilbert space consisting of eigenvectors of the operator.

10.106 ***Spectral Theorem for self-adjoint compact operators***

Suppose T is a self-adjoint compact operator on a Hilbert space V. Then

(a) there is an orthonormal basis of V consisting of eigenvectors of T;

(b) there is a countable set Ω, an orthonormal family $\{e_k\}_{k\in\Omega}$ in V, and a family $\{\alpha_k\}_{k\in\Omega}$ in $\mathbf{R}\setminus\{0\}$ such that

$$Tf = \sum_{k\in\Omega} \alpha_k \langle f, e_k\rangle e_k$$

for every $f \in V$.

Proof Let U denote the span of all the eigenvectors of T. Then U is an invariant subspace for T. Hence $U^\perp$ is also an invariant subspace for T and $T|_{U^\perp}$ is a self-adjoint operator on $U^\perp$ (by 10.102). However, $T|_{U^\perp}$ has no eigenvalues, because all the eigenvectors of T are in U. Because all self-adjoint compact operators on a nonzero Hilbert space have an eigenvalue (by 10.99), this implies that $U^\perp = \{0\}$. Hence $\overline{U} = V$ (by 8.42).

For each eigenvalue α of T, there is an orthonormal basis of $\operatorname{null}(T-\alpha I)$ consisting of eigenvectors corresponding to the eigenvalue α. The union (over all eigenvalues α of T) of all these orthonormal bases is an orthonormal family in V because eigenvectors corresponding to distinct eigenvalues are orthogonal (see 10.57). The previous paragraph tells us that the closure of the span of this orthonormal family is V (here we are using the set itself as the index set). Hence we have an orthonormal basis of V consisting of eigenvectors of T, completing the proof of (a).

By part (a) of this result, there is an orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V and a family $\{\alpha_k\}_{k\in\Gamma}$ in $\mathbf{R}$ such that $Te_k = \alpha_k e_k$ for each $k \in \Gamma$ (even if $\mathbf{F} = \mathbf{C}$, the eigenvalues of T are in $\mathbf{R}$ by 10.49) . Thus if $f \in V$, then

$$Tf = T\Big(\sum_{k\in\Gamma}\langle f, e_k\rangle e_k\Big) = \sum_{k\in\Gamma}\langle f, e_k\rangle Te_k = \sum_{k\in\Gamma}\alpha_k\langle f, e_k\rangle e_k.$$

Letting $\Omega = \{k \in \Gamma : \alpha_k \neq 0\}$, we can rewrite the equation above as

$$Tf = \sum_{k\in\Omega} \alpha_k \langle f, e_k\rangle e_k$$

for every $f \in V$. The set Ω is countable because T has only countably many eigenvalues (by 10.93) and each nonzero eigenvalue can appear only finitely many times in the sum above (by 10.82), completing the proof of (b). ■

A normal compact operator on a nonzero real Hilbert space might have no eigenvalues [consider, for example the normal operator T of counterclockwise rotation by a right angle on $\mathbf{R}^2$ defined by $T(x,y) = (-y,x)$]. However, the next result shows that normal compact operators on ***complex*** Hilbert spaces behave better. The key idea in proving this result is that on a complex Hilbert space, the real and imaginary parts of a normal compact operator are commuting self-adjoint compact operators, which then allows us to apply the Spectral Theorem for self-adjoint compact operators.

10.107 ***Spectral Theorem for normal compact operators***

Suppose T is a compact operator on a complex Hilbert space V. Then there is an orthonormal basis of V consisting of eigenvectors of T if and only if T is normal.

Proof One direction of this result has already been proved as part (b) of 10.103.

To prove the other direction, suppose T is a normal compact operator. We can write

$$T = A + iB,$$

where A and B are self-adjoint operators and, because T is normal, $AB = BA$ (see 10.54). Because $A = (T+T^*)/2$ and $B = (T-T^*)/(2i)$, the operators A and B are both compact.

If $\alpha \in \mathbf{R}$ and $f \in \operatorname{null}(A - \alpha I)$, then

$$(A-\alpha I)(Bf) = A(Bf) - \alpha Bf = B(Af) - \alpha Bf = B\big((A-\alpha I)f\big) = B(0) = 0$$

and thus $Bf \in \operatorname{null}(A-\alpha I)$. Hence $\operatorname{null}(A-\alpha I)$ is an invariant subspace for B.

Applying the Spectral Theorem for self-adjoint compact operators [10.106(a)] to $B|_{\operatorname{null}(A-\alpha I)}$ shows that for each eigenvalue α of A, there is an orthonormal basis of $\operatorname{null}(A-\alpha I)$ consisting of eigenvectors of B. The union (over all eigenvalues α of A) of all these orthonormal bases is an orthonormal family in V (use the set itself as the index set) because eigenvectors of A corresponding to distinct eigenvalues of A are orthogonal (see 10.57). The Spectral Theorem for self-adjoint compact operators [10.106(a)] as applied to A tells us that the closure of the span of this orthonormal family is V. Hence we have an orthonormal basis of V, each of whose elements is an eigenvector of A and an eigenvector of B.

If $f \in V$ is an eigenvector of both A and B, then there exist $\alpha, \beta \in \mathbf{R}$ such that $Af = \alpha f$ and $Bf = \beta f$. Thus $Tf = (A+iB)(f) = (\alpha + \beta i)f$; hence f is an eigenvector of T. Thus the orthonormal basis of V constructed in the previous paragraph is an orthonormal basis consisting of eigenvectors of T, completing the proof. ■

The following example shows the power of the Spectral Theorem for normal compact operators. Finding the eigenvalues and eigenvectors of the normal compact operator $\mathcal{V} - \mathcal{V}^*$ in the next example leads us to an orthonormal basis of $L^2([0,1])$. Easy calculus shows that the family $\{e_k\}_{k\in\mathbf{Z}}$, where e_k is defined as in 10.112, is an orthonormal family in $L^2([0,1])$. The hard part of showing that $\{e_k\}_{k\in\mathbf{Z}}$ is an orthonormal ***basis*** of $L^2([0,1])$ is to show that the closure of the span of this family is $L^2([0,1])$. However, the Spectral Theorem for normal compact operators (10.107) provides this information with no further work required.

10.108 Example *an orthonormal basis of eigenvectors*

Suppose $\mathcal{V}\colon L^2([0,1]) \to L^2([0,1])$ is the Volterra operator defined by

$$(\mathcal{V}f)(x) = \int_0^x f.$$

The operator $\mathcal{V}$ is compact (see the paragraph after the proof of 10.70), but it is not normal. Because $\mathcal{V}$ is compact, so is $\mathcal{V}^*$ (by 10.73). Hence $\mathcal{V} - \mathcal{V}^*$ is compact. Also, $(\mathcal{V} - \mathcal{V}^*)^* = \mathcal{V}^* - \mathcal{V} = -(\mathcal{V} - \mathcal{V}^*)$. Because every operator commutes with its negative, we conclude that $\mathcal{V} - \mathcal{V}^*$ is a compact normal operator. Because we want to apply the Spectral Theorem, for the rest of this example we will take $\mathbf{F} = \mathbf{C}$.

If $f \in L^2([0,1])$ and $x \in [0,1]$, then the formula for $\mathcal{V}^*$ given by 10.16 shows that

10.109 $$\big((\mathcal{V} - \mathcal{V}^*)f\big)(x) = 2\int_0^x f - \int_0^1 f.$$

The right side of the equation above is a continuous function of x whose value at $x = 0$ is the negative of its value at $x = 1$.

Differentiating both sides of the equation above and using the Lebesgue Differentiation Theorem (4.19) shows that

$$\big((\mathcal{V} - \mathcal{V}^*)f\big)'(x) = 2f(x)$$

for almost every $x \in [0,1]$. If $f \in \operatorname{null}(\mathcal{V} - \mathcal{V}^*)$, then differentiating both sides of the equation $(\mathcal{V} - \mathcal{V}^*)f = 0$ shows that $2f(x) = 0$ for almost every $x \in [0,1]$; hence $f = 0$, and we conclude that $\mathcal{V} - \mathcal{V}^*$ is injective (so 0 is not an eigenvalue).

Suppose f is an eigenvector of $\mathcal{V} - \mathcal{V}^*$ with eigenvalue α. Thus f is in the range of $\mathcal{V} - \mathcal{V}^*$, which by 10.109 implies that f is continuous on $[0,1]$, which by 10.109 again implies that f is continuously differentiable on $(0,1)$. Differentiating both sides of the equation $(\mathcal{V} - \mathcal{V}^*)f = \alpha f$ gives

$$2f(x) = \alpha f'(x)$$

for all $x \in (0,1)$. Hence the function whose value at x is $e^{-(2/\alpha)x}f(x)$ has derivative 0 everywhere on $(0,1)$ and thus is a constant function. In other words,

10.110 $$f(x) = ce^{(2/\alpha)x}$$

for some constant $c \neq 0$. Because $f \in \operatorname{range}(V - \mathcal{V}^*)$, we have $f(0) = -f(1)$, which with the equation above implies that there exists $k \in \mathbf{Z}$ such that

10.111 $$2/\alpha = i(2k+1)\pi.$$

Replacing $2/\alpha$ in 10.110 with the value of $2/\alpha$ derived in 10.111 shows that for $k \in \mathbf{Z}$, we should define $e_k \in L^2([0,1])$ by

10.112 $$e_k(x) = e^{i(2k+1)\pi x}.$$

Clearly $\{e_k\}_{k\in\mathbf{Z}}$ is an orthonormal family in $L^2([0,1])$ [the orthogonality can be verified by a straightforward calculation or by using 10.57]. The paragraph above and the Spectral Theorem for compact normal operators (10.107) imply that this orthonormal family is an orthonormal basis of $L^2([0,1])$.

Singular Value Decomposition

The next result provides an important generalization of 10.106(b) to arbitrary compact operators that need not be self-adjoint or normal. This generalization requires two orthonormal families, as compared to the single orthonormal family in 10.106(b).

10.113 *Singular Value Decomposition*

Suppose T is a compact operator on a Hilbert space V. Then there exist a countable set Ω, orthonormal families $\{e_k\}_{k\in\Omega}$ and $\{h_k\}_{k\in\Omega}$ in V, and a family $\{s_k\}_{k\in\Omega}$ of positive numbers such that

10.114
$$Tf = \sum_{k\in\Omega} s_k \langle f, e_k\rangle h_k$$

for every $f \in V$.

Proof If α is an eigenvalue of T^*T, then $(T^*T)f = \alpha f$ for some $f \neq 0$ and

$$\alpha\|f\|^2 = \langle \alpha f, f\rangle = \langle T^*Tf, f\rangle = \langle Tf, Tf\rangle = \|Tf\|^2.$$

Thus $\alpha \geq 0$. Hence all eigenvalues of T^*T are nonnegative.

Apply 10.106(b) and the conclusion of the paragraph above to the self-adjoint compact operator T^*T, getting a countable set Ω, an orthonormal family $\{e_k\}_{k\in\Omega}$ in V, and a family $\{s_k\}_{k\in\Omega}$ of positive numbers (take $s_k = \sqrt{\alpha_k}$) such that

10.115
$$(T^*T)f = \sum_{k\in\Omega} {s_k}^2 \langle f, e_k\rangle e_k$$

for every $f \in V$. The equation above implies that $(T^*T)e_j = {s_j}^2 e_j$ for each $j \in \Omega$.

For $k \in \Omega$, let

$$h_k = \frac{Te_k}{s_k}.$$

For $j, k \in \Omega$, we have

$$\langle h_j, h_k\rangle = \frac{1}{s_j s_k}\langle Te_j, Te_k\rangle = \frac{1}{s_j s_k}\langle T^*Te_j, e_k\rangle = \frac{s_j}{s_k}\langle e_j, e_k\rangle.$$

The equation above implies that $\{h_k\}_{k\in\Omega}$ is an orthonormal family in V.

If $f \in \overline{\operatorname{span}\{e_k\}_{k\in\Omega}}$, then

$$Tf = T\Big(\sum_{k\in\Omega} \langle f, e_k\rangle e_k\Big) = \sum_{k\in\Omega} \langle f, e_k\rangle Te_k = \sum_{k\in\Omega} s_k \langle f, e_k\rangle h_k,$$

showing that 10.114 holds for such f.

If $f \in \big(\overline{\operatorname{span}\{e_k\}_{k\in\Omega}}\big)^\perp$, then 10.115 shows that $(T^*T)f = 0$, which implies that $Tf = 0$ (because $0 = \langle T^*Tf, f\rangle = \|Tf\|^2$); thus both sides of 10.114 are 0.

Hence the two sides of 10.114 agree for f in a closed subspace of V and for f in the orthogonal complement of that closed subspace, which by linearity implies that the two sides of 10.114 agree for all $f \in V$. ∎

An expression of the form 10.114 is called a *singular value decomposition* of the compact operator T. The orthonormal families $\{e_k\}_{k\in\Omega}$ and $\{h_k\}_{k\in\Omega}$ in the singular value decomposition are not uniquely determined by T. However, the positive numbers $\{s_k\}_{k\in\Omega}$ are uniquely determined as positive square roots of positive eigenvalues of T^*T. These positive numbers can be placed in decreasing order (because if there are infinitely many of them, then they form a sequence with limit 0, by 10.93). This procedure leads to the definition of singular values given below.

Suppose T is a compact operator. Recall that the geometric multiplicity of a positive eigenvalue α of T^*T is defined to be $\dim \operatorname{null}(T^*T - \alpha I)$ [see 10.81]. This geometric multiplicity is the number of times that $\sqrt{\alpha}$ appears in the family $\{s_k\}_{k\in\Omega}$ corresponding to a singular value decomposition of T. By 10.82, this geometric multiplicity is finite.

Now we can define the singular values of a compact operator T, where we are careful to repeat the square root of each positive eigenvalue of T^*T as many times as its geometric multiplicity.

10.116 Definition *singular values*

- Suppose T is a compact operator on a Hilbert space. The *singular values* of T, denoted $s_1(T) \geq s_2(T) \geq s_3(T) \geq \cdots$, are the positive square roots of the positive eigenvalues of T^*T, arranged in decreasing order with each singular value s repeated as many times as the geometric multiplicity of s^2 as an eigenvalue of T^*T.

- If T^*T has only finitely many positive eigenvalues, then define $s_n(T) = 0$ for all $n \in \mathbf{Z}^+$ for which $s_n(T)$ is not defined by the first bullet point.

10.117 Example *singular values on a finite-dimensional Hilbert space*

Define $T\colon \mathbf{F}^4 \to \mathbf{F}^4$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$(T^*T)(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Thus the eigenvalues of T^*T are $9, 4, 0$ and

$$\dim(T^*T - 9I) = 2 \quad \text{and} \quad \dim(T^*T - 4I) = 1.$$

Taking square roots of the positive eigenvalues of T^*T and then adjoining an infinite string of 0's shows that the singular values of T are $3 \geq 3 \geq 2 \geq 0 \geq 0 \geq \cdots$.

Note that -3 and 0 are the only eigenvalues of T. Thus in this case, the list of eigenvalues of T did not pick up the number 2 that appears in the definition (and hence the behavior) of T, but the list of singular values of T does include 2.

If T is a compact operator, then the first singular value $s_1(T)$ equals $\|T\|$, as you are asked to verify in Exercise 12.

10.118 **Example** *singular values of $\mathcal{V} - \mathcal{V}^*$*

Let $\mathcal{V}$ denote the Volterra operator and let $T = \mathcal{V} - \mathcal{V}^*$. In Example 10.108, we saw that if e_k is defined by 10.112 then $\{e_k\}_{k\in\mathbf{Z}}$ is an orthonormal basis of $L^2([0,1])$ and

$$Te_k = \frac{2}{i(2k+1)\pi}e_k$$

for each $k \in \mathbf{Z}$, where the eigenvalue shown above corresponding to e_k comes from 10.111. Now 10.56 implies that

$$T^*e_k = \frac{-2}{i(2k+1)\pi}e_k$$

for each $k \in \mathbf{Z}$. Hence

10.119
$$T^*Te_k = \frac{4}{(2k+1)^2\pi^2}e_k$$

for each $k \in \mathbf{Z}$. After taking positive square roots of the eigenvalues, we see that the equation above shows that the singular values of T are

$$\frac{2}{\pi} \geq \frac{2}{\pi} \geq \frac{2}{3\pi} \geq \frac{2}{3\pi} \geq \frac{2}{5\pi} \geq \frac{2}{5\pi} \geq \cdots,$$

where the first two singular values above come from taking $k = -1$ and $k = 0$ in 10.119, the next two singular values above come from taking $k = -2$ and $k = 1$, the next two singular values above come from taking $k = -3$ and $k = 2$, and so on. Each singular value of T appears twice in the list of singular values above because each eigenvalue of T^*T has geometric multiplicity 2.

For $n \in \mathbf{Z}^+$, the singular value $s_n(T)$ of a compact operator T tells us how well we can approximate T by operators whose range has dimension less than n (see Exercise 15).

The next result makes an important connection between $K \in L^2(\mu \times \mu)$ and the singular values of the integral operator associated with K.

10.120 ***sum of squares of singular values of integral operator***

Suppose μ is a σ-finite measure and $K \in L^2(\mu \times \mu)$. Then

$$\|K\|^2_{L^2(\mu\times\mu)} = \sum_{n=1}^{\infty} \big(s_n(\mathcal{I}_K)\big)^2.$$

Proof Consider a singular value decomposition

10.121
$$\mathcal{I}_K(f) = \sum_{k\in\Omega} s_k\langle f, e_k\rangle h_k$$

of the compact operator $\mathcal{I}_K$. Extend $\{e_j\}_{j\in\Omega}$ to an orthonormal basis $\{e_j\}_{j\in\Gamma}$ of $L^2(\mu)$, and extend $\{h_k\}_{k\in\Omega}$ to an orthonormal basis $\{h_k\}_{k\in\Gamma'}$ of $L^2(\mu)$.

Let X denote the set on which the measure μ lives. For $j \in \Gamma$ and $k \in \Gamma'$, define $g_{j,k} \colon X \times X \to \mathbf{F}$ by

$$g_{j,k}(x,y) = \overline{e_j(y)} h_k(x).$$

Then $\{g_{j,k}\}_{j\in\Gamma, k\in\Gamma'}$ is an orthonormal basis of $L^2(\mu\times\mu)$, as you should verify. Thus

$$\begin{aligned}
\|K\|^2_{L^2(\mu\times\mu)} &= \sum_{j\in\Gamma, k\in\Gamma'} |\langle K, g_{j,k}\rangle|^2 \\
&= \sum_{j\in\Gamma, k\in\Gamma'} \left| \int\int K(x,y) e_j(y) \overline{h_k(x)}\, d\mu(y)\, d\mu(x) \right|^2 \\
&= \sum_{j\in\Gamma, k\in\Gamma'} \left| \int (\mathcal{I}_K e_j)(x) \overline{h_k(x)}\, d\mu(x) \right|^2 \\
\textbf{10.122} \qquad &= \sum_{j\in\Omega, k\in\Gamma'} \left| \int s_j h_j(x) \overline{h_k(x)}\, d\mu(x) \right|^2 \\
\textbf{10.123} \qquad &= \sum_{j\in\Omega} {s_j}^2 \\
&= \sum_{n=1}^{\infty} \bigl(s_n(\mathcal{I}_K)\bigr)^2,
\end{aligned}$$

where 10.122 holds because 10.121 shows that $\mathcal{I}_K e_j = s_j h_j$ for $j \in \Omega$ and $\mathcal{I}_K e_j = 0$ for $j \in \Gamma \setminus \Omega$; 10.123 holds because $\{h_k\}_{k\in\Gamma'}$ is an orthonormal family. ■

Now we can give a spectacular application of the previous result.

10.124 Example $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

Define $K \colon [0,1] \times [0,1] \to \mathbf{R}$ by

$$K(x,y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -1 & \text{if } x < y. \end{cases}$$

Letting μ be Lebesgue measure on $[0,1]$, we note that $\mathcal{I}_K$ is the normal compact operator $\mathcal{V} - \mathcal{V}^*$ examined in Example 10.118.

Clearly $\|K\|_{L^2(\mu\times\mu)} = 1$. Using the list of singular values for $\mathcal{I}_K$ obtained in Example 10.118, the formula in 10.120 tells us that

$$1 = 2\sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}.$$

Thus

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

EXERCISES 10D

1 Prove that if T is a compact operator on a nonzero Hilbert space, then $\|T\|^2$ is an eigenvalue of T^*T.

2 Prove that if T is a self-adjoint operator on a nonzero Hilbert space V, then

$$\|T\| = \sup\{|\langle Tf, f\rangle| : f \in V \text{ and } \|f\| = 1\}.$$

3 Suppose T is a bounded operator on a Hilbert space V and U is a closed subspace of V. Prove that the following are equivalent:

(a) U is an invariant subspace for T.

(b) $U^\perp$ is an invariant subspace for T^*.

(c) $TP_U = P_U T P_U$.

4 Suppose T is a bounded operator on a Hilbert space V and U is a closed subspace of V. Prove that the following are equivalent:

(a) U and $U^\perp$ are invariant subspaces for T.

(b) U and $U^\perp$ are invariant subspaces for T^*.

(c) $TP_U = P_U T$.

5 Suppose T is a bounded operator on a nonseparable normed vector space V. Prove that T has a closed invariant subspace other than $\{0\}$ and V.

6 Suppose T is an operator on a Banach space V with dimension greater than 2. Prove that T has an invariant subspace other than $\{0\}$ and V.
[*For this exercise, T is not assumed to be bounded and the invariant subspace is not required to be closed.*]

7 Suppose T is a self-adjoint compact operator on a Hilbert space that has only finitely many distinct eigenvalues. Prove that T has finite-dimensional range.

8 (a) Prove that if T is a self-adjoint compact operator on a Hilbert space, then there exists a self-adjoint compact operator S such that $S^3 = T$.

(b) Prove that if T is a normal compact operator on a complex Hilbert space, then there exists a normal compact operator S such that $S^2 = T$.

9 Suppose T is a compact normal operator on a nonzero Hilbert space V. Prove that there is a subspace of V with dimension 1 or 2 that is an invariant subspace for T.
[*If* $\mathbf{F} = \mathbf{C}$, *the desired result follows immediately from the Spectral Theorem for compact normal operators. Thus you can assume that* $\mathbf{F} = \mathbf{R}$.]

10 Suppose T is a self-adjoint compact operator on a Hilbert space and $\|T\| \le \frac{1}{4}$. Prove that there exists a self-adjoint compact operator S such that $S^2 + S = T$.

11 For $k \in \mathbf{Z}$, define $g_k \in L^2((-\pi,\pi])$ and $h_k \in L^2((-\pi,\pi])$ by

$$g_k(t) = \frac{1}{\sqrt{2\pi}} e^{it/2} e^{ikt} \quad \text{and} \quad h_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt};$$

here we are assuming that $\mathbf{F} = \mathbf{C}$.

(a) Use the conclusion of Example 10.108 to show that $\{g_k\}_{k\in\mathbf{Z}}$ is an orthonormal basis of $L^2((-\pi,\pi])$.

(b) Use the result in part (a) to show that $\{h_k\}_{k\in\mathbf{Z}}$ is an orthonormal basis of $L^2((-\pi,\pi])$.

(c) Use the result in part (b) to show that the orthonormal family in the third bullet point of Example 8.51 is an orthonormal basis of $L^2((-\pi,\pi])$.

12 Suppose T is a compact operator on a Hilbert space. Prove that $s_1(T) = \|T\|$.

13 Suppose T is a compact operator on a Hilbert space and $n \in \mathbf{Z}^+$. Prove that $\dim \operatorname{range} T < n$ if and only if $s_n(T) = 0$.

14 Suppose T is a compact operator on a Hilbert space V with singular value decomposition

$$Tf = \sum_{k=1}^{\infty} s_k(T)\langle f, e_k\rangle h_k$$

for all $f \in V$. For $n \in \mathbf{Z}^+$, define $T_n \colon V \to V$ by

$$T_n f = \sum_{k=1}^{n} s_k(T)\langle f, e_k\rangle h_k.$$

Prove that $\lim_{n\to\infty} \|T - T_n\| = 0$.
[*This exercise gives another proof, in addition to the proof suggested by Exercise 15 in Section 10C, that an operator on a Hilbert space is compact if and only if it is the limit of bounded operators with finite-dimensional range.*]

15 Suppose T is a compact operator on a Hilbert space V and $n \in \mathbf{Z}^+$. Prove that

$$\inf\{\|T - S\| : S \in \mathcal{B}(V) \text{ and } \dim \operatorname{range} S < n\} = s_n(T).$$

16 Suppose T is a compact operator on a Hilbert space V and $n \in \mathbf{Z}^+$. Prove that

$$s_n(T) = \inf\{\|T|_{U^\perp}\| : U \text{ is a subspace of } V \text{ with } \dim U < n\}.$$

17 Suppose T is a compact operator on a Hilbert space with singular value decomposition

$$Tf = \sum_{k\in\Omega} s_k\langle f, e_k\rangle h_k$$

for all $f \in V$. Prove that

$$T^* f = \sum_{k\in\Omega} s_k\langle f, h_k\rangle e_k$$

for all $f \in V$.

18 Suppose that T is an operator on a finite-dimensional Hilbert space V with $\dim V = n$.

(a) Prove that T is invertible if and only if $s_n(T) \neq 0$.

(b) Suppose T is invertible and T has a singular value decomposition

$$Tf = s_1(T)\langle f, e_1\rangle h_1 + \cdots + s_n(T)\langle f, e_n\rangle h_n$$

for all $f \in V$. Show that

$$T^{-1}f = \frac{\langle f, h_1\rangle}{s_1(T)} e_1 + \cdots + \frac{\langle f, h_n\rangle}{s_n(T)} e_n$$

for all $f \in V$.

19 Suppose T is a compact operator on a Hilbert space V. Prove that

$$\sum_{k\in\Gamma} \|Te_k\|^2 = \sum_{n=1}^{\infty} \big(s_n(T)\big)^2$$

for every orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V.

20 Use the result of Example 10.124 to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

21 Suppose T is a normal compact operator. Prove that the following are equivalent:

- range T is finite-dimensional.
- $\operatorname{sp}(T)$ is a finite set.
- $s_n(T) = 0$ for some $n \in \mathbf{Z}$.

22 Find the singular values of the Volterra operator.
[*Your answer, when combined with Exercise 12, should show that the norm of the Volterra operator is* $\frac{2}{\pi}$. *This appearance of* π *can be surprising because the definition of the Volterra operator does not involve* π.]

Chapter 11
Fourier Analysis

This chapter uses Hilbert space theory to motivate the introduction of Fourier coefficients and Fourier series. The classical setting applies these concepts to functions defined on bounded intervals of the real line. However, the theory becomes easier and cleaner when we instead use a modern approach by considering functions defined on the unit circle of the complex plane.

The first section of this chapter shows how consideration of Fourier series leads us to harmonic functions and a solution to the Dirichlet problem. In the second section of this chapter, convolution becomes a major tool for the L^p theory.

The third section of this chapter changes the context to functions defined on the real line. Many of the techniques introduced in the first two sections of the chapter transfer easily to provide results about the Fourier transform on the real line. The highlights of our treatment of the Fourier transform are the Fourier Inversion Formula and the extension of the Fourier transform to a unitary operator on $L^2(\mathbf{R})$.

The vast field of Fourier analysis cannot be completely covered in a single chapter. Thus this chapter gives readers just a taste of the subject. Readers who go on from this chapter to one of the many book-length treatments of Fourier analysis will then already be familiar with the terminology and techniques of the subject.



The Giza pyramids, near where the Battle of Pyramids took place in 1798 during Napoleon's invasion of Egypt. Joseph Fourier (1768–1830) was one of the scientific advisors to Napoleon in Egypt. While in Egypt as part of Napoleon's invading force, Fourier began thinking about the mathematical theory of heat propagation, which eventually led to what we now call Fourier series and the Fourier transform.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_11

11A Fourier Series and Poisson Integral

Fourier Coefficients and Riemann–Lebesgue Lemma

For $k \in \mathbf{Z}$, suppose $e_k \colon (-\pi, \pi] \to \mathbf{R}$ is defined by

11.1
$$e_k(t) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(kt) & \text{if } k > 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0, \\ \frac{1}{\sqrt{\pi}} \cos(kt) & \text{if } k < 0. \end{cases}$$

The classical theory of Fourier series features $\{e_k\}_{k \in \mathbf{Z}}$ as an orthonormal basis of $L^2\big((-\pi, \pi]\big)$. The trigonometric formulas displayed in Exercise 1 in Section 8C can be used to show that $\{e_k\}_{k \in \mathbf{Z}}$ is indeed an orthonormal family in $L^2\big((-\pi, \pi]\big)$.

To show that $\{e_k\}_{k \in \mathbf{Z}}$ is an orthonormal basis of $L^2\big((-\pi, \pi]\big)$ requires more work. One slick possibility is to note that the Spectral Theorem for compact operators produces orthonormal bases; an appropriate choice of a compact normal operator can then be used to show that $\{e_k\}_{k \in \mathbf{Z}}$ is an orthonormal basis of $L^2\big((-\pi, \pi]\big)$ [see Exercise 11(c) in Section 10D].

In this chapter we take a cleaner approach to Fourier series by working on the unit circle in the complex plane instead of on the interval $(-\pi, \pi]$. The map

11.2
$$t \mapsto e^{it} = \cos t + i \sin t$$

can be used to identify the interval $(-\pi, \pi]$ with the unit circle; thus the two approaches are equivalent. However, the calculations are easier in the unit circle context. In addition, we will see that the unit circle context provides the huge benefit of making a connection with harmonic functions.

We begin by introducing notation for the open unit disk and the unit circle in the complex plane.

11.3 Definition $\mathbf{D}$; $\partial\mathbf{D}$

- $\mathbf{D}$ denotes the open unit disk in the complex plane:
$$\mathbf{D} = \{w \in \mathbf{C} : |w| < 1\}.$$
- $\partial\mathbf{D}$ is the unit circle in the complex plane:
$$\partial\mathbf{D} = \{z \in \mathbf{C} : |z| = 1\}.$$

The function given in 11.2 is a one-to-one map of $(-\pi, \pi]$ onto $\partial\mathbf{D}$. We use this map to define a σ-algebra on $\partial\mathbf{D}$ by transferring the Borel subsets of $(-\pi, \pi]$ to subsets of $\partial\mathbf{D}$ that we will call the *measurable subsets* of $\partial\mathbf{D}$. We also transfer Lebesgue measure on the Borel subsets of $(\pi, \pi]$ to a measure called σ on the measurable subsets of $\partial\mathbf{D}$, except that for convenience we normalize by dividing by 2π so that the measure of $\partial\mathbf{D}$ is 1 rather than 2π. We are now ready to give the formal definitions.

11.4 Definition ***measurable subsets of $\partial\mathbf{D}$; σ***

- A subset E of $\partial\mathbf{D}$ is *measurable* if $\{t \in (-\pi, \pi] : e^{it} \in E\}$ is a Borel subset of $\mathbf{R}$.
- σ is the measure on the measurable subsets of $\partial\mathbf{D}$ obtained by transferring Lebesgue measure from $(-\pi, \pi]$ to $\partial\mathbf{D}$, normalized so that $\sigma(\partial\mathbf{D}) = 1$. In other words, if $E \subset \partial\mathbf{D}$ is measurable, then

$$\sigma(E) = \frac{|\{t \in (-\pi, \pi] : e^{it} \in E\}|}{2\pi}.$$

Our definition of the measure σ on $\partial\mathbf{D}$ allows us to transfer integration on $\partial\mathbf{D}$ to the familiar context of integration on $(-\pi, \pi]$. Specifically,

$$\int_{\partial\mathbf{D}} f\, d\sigma = \int_{\partial\mathbf{D}} f(z)\, d\sigma(z) = \int_{-\pi}^{\pi} f(e^{it})\, \frac{dt}{2\pi}$$

for all measurable functions $f : \partial\mathbf{D} \to \mathbf{C}$ such that any of these integrals is defined.

Throughout this chapter, we assume that the scalar field $\mathbf{F}$ is the complex field $\mathbf{C}$. Furthermore, $L^p(\partial\mathbf{D})$ is defined as follows.

11.5 Definition $L^p(\partial\mathbf{D})$

For $1 \le p \le \infty$, define $L^p(\partial\mathbf{D})$ to mean the complex version ($\mathbf{F} = \mathbf{C}$) of $L^p(\sigma)$.

Note that if $z = e^{it}$ for some $t \in \mathbf{R}$, then $\overline{z} = e^{-it} = \frac{1}{z}$ and $z^n = e^{int}$ and $\overline{z^n} = e^{-int}$ for all $n \in \mathbf{Z}$. These observations make the proof of the next result much simpler than the proof of the corresponding result for the trigonometric family defined by 11.1.

In the statement of the next result, z^n means the function on $\partial\mathbf{D}$ defined by $z \mapsto z^n$.

11.6 ***orthonormal family in*** $L^2(\partial\mathbf{D})$

$\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal family in $L^2(\partial\mathbf{D})$.

Proof If $n \in \mathbf{Z}$, then

$$\langle z^n, z^n \rangle = \int_{\partial\mathbf{D}} |z^n|^2\, d\sigma(z) = \int_{\partial\mathbf{D}} 1\, d\sigma = 1.$$

If $m, n \in \mathbf{Z}$ with $m \neq n$, then

$$\langle z^m, z^n \rangle = \int_{-\pi}^{\pi} e^{imt} e^{-int}\, \frac{dt}{2\pi} = \int_{-\pi}^{\pi} e^{i(m-n)t}\, \frac{dt}{2\pi} = \frac{e^{i(m-n)t}}{i(m-n)2\pi}\Big]_{t=-\pi}^{t=\pi} = 0,$$

as desired. ■

In the next section, we improve the result above by showing that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$ (see 11.30).

Hilbert space theory tells us that if f is in the closure in $L^2(\partial\mathbf{D})$ of $\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}$, then

$$f = \sum_{n\in\mathbf{Z}} \langle f, z^n\rangle z^n,$$

where the infinite sum above converges as an unordered sum in the norm of $L^2(\partial\mathbf{D})$ (see 8.58). The inner product $\langle f, z^n\rangle$ above equals

$$\int_{\partial\mathbf{D}} f(z)\overline{z^n}\, d\sigma(z).$$

Because $|z^n| = 1$ for every $z \in \partial\mathbf{D}$, the integral above makes sense not only for $f \in L^2(\partial\mathbf{D})$ but also for f in the larger space $L^1(\partial\mathbf{D})$. Thus we make the following definition.

11.7 Definition ***Fourier coefficient***

Suppose $f \in L^1(\partial\mathbf{D})$.

- For $n \in \mathbf{Z}$, the nth *Fourier coefficient* of f is denoted $\hat{f}(n)$ and is defined by

$$\hat{f}(n) = \int_{\partial\mathbf{D}} f(z)\overline{z^n}\, d\sigma(z) = \int_{-\pi}^{\pi} f(e^{it})e^{-int}\,\frac{dt}{2\pi}.$$

- The *Fourier series* of f is the formal sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n.$$

As we will see, Fourier analysis helps describe the sense in which the Fourier series of f represents f.

11.8 Example *Fourier coefficients*

- Suppose h is an analytic function on an open set that contains $\overline{D}$. Then h has a power series representation

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the sum on the right converges uniformly on $\overline{D}$ to h. Because uniform convergence on $\partial\mathbf{D}$ implies convergence in $L^2(\partial\mathbf{D})$, 8.58(b) and 11.6 now imply that

$$(h|_{\partial\mathbf{D}})\hat{}\,(n) = \begin{cases} a_n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0 \end{cases}$$

for all $n \in \mathbf{Z}$. In other words, for functions analytic on an open set containing $\overline{D}$, the Fourier series is the same as the Taylor series.

- Suppose $f\colon \partial\mathbf{D} \to \mathbf{R}$ is defined by

$$f(z) = \frac{1}{|3-z|^2}.$$

Then for $z \in \partial\mathbf{D}$ we have

$$\begin{aligned} f(z) &= \frac{1}{(3-z)(3-\overline{z})} \\ &= \frac{1}{8}\Big(\frac{z}{3-z} + \frac{3}{3-\overline{z}}\Big) \\ &= \frac{1}{8}\Big(\frac{\frac{z}{3}}{1-\frac{z}{3}} + \frac{1}{1-\frac{\overline{z}}{3}}\Big) \\ &= \frac{1}{8}\Big(\frac{z}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \sum_{n=0}^{\infty}\frac{(\overline{z})^n}{3^n}\Big) \\ &= \frac{1}{8}\sum_{n=-\infty}^{\infty}\frac{z^n}{3^{|n|}}, \end{aligned}$$

where the infinite sums above converge uniformly on $\partial\mathbf{D}$. Thus we see that

$$\hat{f}(n) = \frac{1}{8}\cdot\frac{1}{3^{|n|}}$$

for all $n \in \mathbf{Z}$.

We begin with some simple algebraic properties of Fourier coefficients, whose proof is left to the reader.

11.9 ***algebraic properties of Fourier coefficients***

Suppose $f, g \in L^1(\partial\mathbf{D})$ and $n \in \mathbf{Z}$. Then

(a) $(f+g)\hat{\ }(n) = \hat{f}(n) + \hat{g}(n)$;

(b) $(\alpha f)\hat{\ }(n) = \alpha\hat{f}(n)$ for all $\alpha \in \mathbf{C}$;

(c) $|\hat{f}(n)| \leq \|f\|_1$.

Parts (a) and (b) above could be restated by saying that for each $n \in \mathbf{Z}$, the function $f \mapsto \hat{f}(n)$ is a linear functional from $L^1(\partial\mathbf{D})$ to $\mathbf{C}$. Part (c) could be restated by saying that this linear functional has norm at most 1.

Part (c) above implies that the set of Fourier coefficients $\{\hat{f}(n)\}_{n\in\mathbf{Z}}$ is bounded for each $f \in L^1(\partial\mathbf{D})$. The Fourier coefficients of the functions in Example 11.8 have the stronger property that $\lim_{n\to\pm\infty}\hat{f}(n) = 0$. The next result shows that this stronger conclusion holds for all functions in $L^1(\partial\mathbf{D})$.

11.10 *Riemann–Lebesgue Lemma*

Suppose $f \in L^1(\partial\mathbf{D})$. Then $\lim\limits_{n\to\pm\infty} \hat{f}(n) = 0$.

Proof Suppose $\varepsilon > 0$. There exists $g \in L^2(\partial\mathbf{D})$ such that $\|f - g\|_1 < \varepsilon$ (by 3.44). By 11.6 and Bessel's inequality (8.57), we have

$$\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 \leq \|g\|_2^2 < \infty.$$

Thus there exists $M \in \mathbf{Z}^+$ such that $|\hat{g}(n)| < \varepsilon$ for all $n \in \mathbf{Z}$ with $|n| \geq M$. Now if $n \in \mathbf{Z}$ and $|n| \geq M$, then

$$\begin{aligned} |\hat{f}(n)| &\leq |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n)| \\ &< |(f-g)\hat{\ }(n)| + \varepsilon \\ &\leq \|f - g\|_1 + \varepsilon \\ &< 2\varepsilon. \end{aligned}$$

Thus $\lim\limits_{n\to\pm\infty} \hat{f}(n) = 0$. ■

Poisson Kernel

Suppose $f \colon \partial\mathbf{D} \to \mathbf{C}$ is continuous and $z \in \partial\mathbf{D}$. For this fixed $z \in \partial\mathbf{D}$, the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$$

is a series of complex numbers. It would be nice if $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$, but this is not necessarily true because the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ might not converge, as you can see in Exercise 11.

Various techniques exist for trying to assign some meaning to a series of complex numbers that does not converge. In one such technique, called *Abel summation*, the nth-term of the series is multiplied by r^n and then the limit is taken as $r \uparrow 1$. For example, if the nth-term of the divergent series

$$1 - 1 + 1 - 1 + \cdots$$

is multiplied by r^n for $r \in [0, 1)$, we get a convergent series whose sum equals $\frac{r}{1+r}$. Taking the limit of this sum as $r \uparrow 1$ then gives $\frac{1}{2}$ as the value of the Abel sum of the series above.

The next definition can be motivated by applying a similar technique to the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$. Here we have a series of complex numbers whose terms are indexed by $\mathbf{Z}$ rather than by $\mathbf{Z}^+$. Thus we use $r^{|n|}$ rather than r^n because we want these multipliers to have limit 0 as $n \to \pm\infty$ for each $r \in [0, 1)$ (and to have limit 1 as $r \uparrow 1$ for each $n \in \mathbf{Z}$).

11.11 Definition $\mathcal{P}_r f$

For $f \in L^1(\partial\mathbf{D})$ and $0 \le r < 1$, define $\mathcal{P}_r f \colon \partial\mathbf{D} \to \mathbf{C}$ by

$$(\mathcal{P}_r f)(z) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) z^n.$$

No convergence problems arise in the series above because

$$\bigl|r^{|n|} \hat{f}(n) z^n\bigr| \le \|f\|_1 \, r^{|n|}$$

for each $z \in \partial\mathbf{D}$, which implies that

$$\sum_{n=-\infty}^{\infty} |r^{|n|} \hat{f}(n) z^n| \le \|f\|_1 \frac{1+r}{1-r} < \infty.$$

Thus for each $r \in [0,1)$, the partial sums of the series above converge uniformly on $\partial\mathbf{D}$, which implies that $\mathcal{P}_r f$ is a continuous function from $\partial\mathbf{D}$ to $\mathbf{C}$ (for $r = 0$ and $n = 0$, interpret the expression 0^0 to be 1).

Let's unravel the formula in 11.11. If $f \in L^1(\partial\mathbf{D})$, $0 \le r < 1$, and $z \in \partial\mathbf{D}$, then

$$\begin{aligned}(\mathcal{P}_r f)(z) &= \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) z^n \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \int_{\partial\mathbf{D}} f(w) \overline{w^n} \, d\sigma(w) z^n \\ &= \int_{\partial\mathbf{D}} f(w) \Bigl(\sum_{n=-\infty}^{\infty} r^{|n|} (z\overline{w})^n \Bigr) \, d\sigma(w), \qquad (11.12)\end{aligned}$$

where interchanging the sum and integral above is justified by the uniform convergence of the series on $\partial\mathbf{D}$. To evaluate the sum in parentheses in the last line above, let $\zeta \in \partial\mathbf{D}$ (think of $\zeta = z\overline{w}$ in the formula above). Thus $(\zeta)^{-n} = (\overline{\zeta})^n$ and

$$\begin{aligned}\sum_{n=-\infty}^{\infty} r^{|n|} \zeta^n &= \sum_{n=0}^{\infty} (r\zeta)^n + \sum_{n=1}^{\infty} (r\overline{\zeta})^n \\ &= \frac{1}{1 - r\zeta} + \frac{r\overline{\zeta}}{1 - r\overline{\zeta}} \\ &= \frac{(1 - r\overline{\zeta}) + (1 - r\zeta) r\overline{\zeta}}{|1 - r\zeta|^2} \\ &= \frac{1 - r^2}{|1 - r\zeta|^2}. \qquad (11.13)\end{aligned}$$

Motivated by the formula above, we now make the following definition. Notice that 11.11 uses calligraphic $\mathcal{P}$, while the next definition uses italic P.

11.14 Definition ***$P_r(\zeta)$; Poisson kernel***

- For $0 \le r < 1$, define $P_r \colon \partial\mathbf{D} \to (0, \infty)$ by
$$P_r(\zeta) = \frac{1 - r^2}{|1 - r\zeta|^2}.$$
- The family of functions $\{P_r\}_{r \in [0,1)}$ is called the *Poisson kernel* on $\mathbf{D}$.

Combining 11.12 and 11.13 now gives the following result.

11.15 ***integral formula for $\mathcal{P}_r f$***

If $f \in L^1(\partial\mathbf{D})$, $0 \le r < 1$, and $z \in \partial\mathbf{D}$, then

$$(\mathcal{P}_r f)(z) = \int_{\partial\mathbf{D}} f(w) P_r(z\overline{w})\, d\sigma(w) = \int_{\partial\mathbf{D}} f(w) \frac{1 - r^2}{|1 - rz\overline{w}|^2}\, d\sigma(w).$$

The terminology *approximate identity* is sometimes used to describe the three properties for the Poisson kernel given in the next result.

11.16 ***properties of P_r***

(a) $P_r(\zeta) > 0$ for all $r \in [0, 1)$ and all $\zeta \in \partial\mathbf{D}$.

(b) $\displaystyle\int_{\partial\mathbf{D}} P_r(\zeta)\, d\sigma(\zeta) = 1$ for each $r \in [0, 1)$.

(c) $\displaystyle\lim_{r \uparrow 1} \int_{\{\zeta \in \partial\mathbf{D} : |1 - \zeta| \ge \delta\}} P_r(\zeta)\, d\sigma(\zeta) = 0$ for each $\delta > 0$.

Proof Part (a) follows immediately from the definition of $P_r(\zeta)$ given in 11.14.

Part (b) follows from integrating the series representation for P_r given by 11.13 termwise and noting that

$$\int_{\partial\mathbf{D}} \zeta^n\, d\sigma(\zeta) = \int_{-\pi}^{\pi} e^{int}\, \frac{dt}{2\pi} = \frac{e^{int}}{in2\pi}\bigg]_{t=-\pi}^{t=\pi} = 0 \text{ for all } n \in \mathbf{Z} \setminus \{0\};$$

for $n = 0$, we have $\int_{\partial\mathbf{D}} \zeta^n\, d\sigma(\zeta) = 1$.

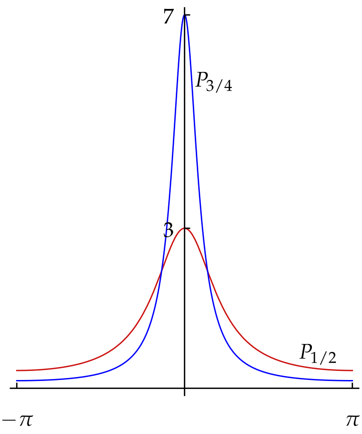
To prove part (c), suppose $\delta > 0$. If $\zeta \in \partial\mathbf{D}$, $|1 - \zeta| \ge \delta$, and $1 - r < \frac{\delta}{2}$, then

$$\begin{aligned} |1 - r\zeta| &= |1 - \zeta - (r - 1)\zeta| \\ &\ge |1 - \zeta| - (1 - r) \\ &> \tfrac{\delta}{2}. \end{aligned}$$

Thus as $r \uparrow 1$, the denominator in the definition of $P_r(\zeta)$ is uniformly bounded away from 0 on $\{\zeta \in \partial\mathbf{D} : |1 - \zeta| \ge \delta\}$ and the numerator goes to 0. Thus the integral of P_r over $\{\zeta \in \partial\mathbf{D} : |1 - \zeta| \ge \delta\}$ goes to 0 as $r \uparrow 1$. ∎

Here is the intuition behind the proof of the next result: Parts (a) and (b) of the previous result and 11.15 mean that $(\mathcal{P}_r f)(z)$ is a weighted average of f. Part (c) of the previous result says that for r close to 1, most of the weight in this weighted average is concentrated near z. Thus $(\mathcal{P}_r f)(z) \to f(z)$ as $r \uparrow 1$.

The figure here transfers the context from $\partial\mathbf{D}$ to $(-\pi, \pi]$. The area under both curves is 2π [corresponding to 11.16(b)] and $P_r(e^{it})$ becomes more concentrated near $t = 0$ as $r \uparrow 1$ [corresponding to 11.16(c)]. See Exercise 3 for the formula for $P_r(e^{it})$.



The graphs of $P_{\frac{1}{2}}(e^{it})$ [*red*] *and* $P_{\frac{3}{4}}(e^{it})$ [*blue*] *on* $(-\pi, \pi]$.

One more ingredient is needed for the next proof: If $h \in L^1(\partial\mathbf{D})$ and $z \in \partial\mathbf{D}$, then

11.17 $$\int_{\partial\mathbf{D}} h(z\overline{w})\, d\sigma(w) = \int_{\partial\mathbf{D}} h(\zeta)\, d\sigma(\zeta).$$

The equation above holds because the measure σ is rotation and reflection invariant. In other words, $\sigma(\{w \in \partial\mathbf{D} : h(z\overline{w}) \in E\}) = \sigma(\{\zeta \in \partial\mathbf{D} : h(\zeta) \in E\})$ for all measurable $E \subset \partial\mathbf{D}$.

11.18 ***if f is continuous, then*** $\lim_{r\uparrow 1} \|f - \mathcal{P}_r f\|_\infty = 0$

Suppose $f : \partial\mathbf{D} \to \mathbf{C}$ is continuous. Then $\mathcal{P}_r f$ converges uniformly to f on $\partial\mathbf{D}$ as $r \uparrow 1$.

Proof Suppose $\varepsilon > 0$. Because f is uniformly continuous on $\partial\mathbf{D}$, there exists $\delta > 0$ such that

$$|f(z) - f(w)| < \varepsilon \text{ for all } z, w \in \partial\mathbf{D} \text{ with } |z - w| < \delta.$$

If $z \in \partial\mathbf{D}$, then

$$\begin{aligned}
|f(z) - (\mathcal{P}_r f)(z)| &= \left| f(z) - \int_{\partial\mathbf{D}} f(w) P_r(z\overline{w})\, d\sigma(w) \right| \\
&= \left| \int_{\partial\mathbf{D}} \big(f(z) - f(w)\big) P_r(z\overline{w})\, d\sigma(w) \right| \\
&\le \varepsilon \int_{\{w \in \partial\mathbf{D} : |z-w| < \delta\}} P_r(z\overline{w})\, d\sigma(w) \\
&\qquad + 2\|f\|_\infty \int_{\{w \in \partial\mathbf{D} : |z-w| \ge \delta\}} P_r(z\overline{w})\, d\sigma(w) \\
&\le \varepsilon + 2\|f\|_\infty \int_{\{\zeta \in \partial\mathbf{D} : |1-\zeta| \ge \delta\}} P_r(\zeta)\, d\sigma(\zeta),
\end{aligned}$$

where we have used 11.17, 11.16(a), and 11.16(b); the last line uses the equality $|z - w| = |1 - \zeta|$, which holds when $\zeta = z\overline{w}$. Now 11.16(c) shows that the value of the last integral above has uniform (with respect to $z \in \partial\mathbf{D}$) limit 0 as $r \uparrow 1$. ∎

Solution to Dirichlet Problem on Disk

As a bonus to our investigation into Fourier series, the previous result provides the solution to the Dirichlet problem on the unit disk. To state the Dirichlet problem, we first need a few definitions. As usual, we identify $\mathbf{C}$ with $\mathbf{R}^2$. Thus for $x, y \in \mathbf{R}$, we can think of $w = x + yi \in \mathbf{C}$ or $w = (x, y) \in \mathbf{R}^2$. Hence

$$\mathbf{D} = \{w \in \mathbf{C} : |w| < 1\} = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}.$$

For a function $f\colon G \to \mathbf{C}$ on an open subset G of $\mathbf{C}$ (or an open subset G of $\mathbf{R}^2$), the partial derivatives $D_1 f$ and $D_2 f$ are defined as in 5.46 except that now we allow f to be a complex-valued function. Clearly $D_j f = D_j(\operatorname{Re} f) + iD_j(\operatorname{Im} f)$ for $j = 1, 2$.

11.19 Definition ***harmonic function***

A function $u\colon G \to \mathbf{C}$ on an open subset G of $\mathbf{R}^2$ is called *harmonic* if

$$\big(D_1(D_1 f)\big)(w) + \big(D_2(D_2 f)\big)(w) = 0$$

for all $w \in G$. The left side of the equation above is called the *Laplacian* of f at w and is often denoted by $(\Delta f)(w)$.

11.20 Example *harmonic functions*

- If $f\colon G \to \mathbf{C}$ is an analytic function on an open set $G \subset \mathbf{C}$, then the functions $\operatorname{Re} f$, $\operatorname{Im} f$, f, and $\overline{f}$ are all harmonic functions on G, as is usually discussed near the beginning of a course on complex analysis.
- If $\zeta \in \partial\mathbf{D}$, then the function
$$w \mapsto \frac{1 - |w|^2}{|1 - \overline{\zeta} w|^2}$$
is harmonic on $\mathbf{C} \setminus \{\zeta\}$ (see Exercise 7).
- The function $u\colon \mathbf{C} \setminus \{0\} \to \mathbf{R}$ defined by $u(w) = \log|w|$ is harmonic on $\mathbf{C} \setminus \{0\}$, as you should verify. However, there does not exist a function f analytic on $\mathbf{C} \setminus \{0\}$ such that $u = \operatorname{Re} f$.

The Dirichlet problem asks to extend a continuous function on the boundary of an open subset of $\mathbf{R}^2$ to a function that is harmonic on the open set and continuous on the closure of the open set. Here is a more formal statement:

11.21 **Dirichlet problem on G:** Suppose $G \subset \mathbf{R}^2$ is an open set and $f\colon \partial G \to \mathbf{C}$ is a continuous function. Find a continuous function $u\colon \overline{G} \to \mathbf{C}$ such that $u|_G$ is harmonic and $u|_{\partial G} = f$.

For some open sets $G \subset \mathbf{R}^2$, there exist continuous functions f on ∂G whose Dirichlet problem has no solution. However, the situation on the open unit disk $\mathbf{D}$ is much nicer, as we will soon see.

The function u defined in the result below is called the *Poisson integral* of f on $\mathbf{D}$.

11.22 *Poisson integral is harmonic*

Suppose $f \in L^1(\partial\mathbf{D})$. Define $u\colon \mathbf{D} \to \mathbf{C}$ by

$$u(rz) = (\mathcal{P}_r f)(z)$$

for $r \in [0,1)$ and $z \in \partial\mathbf{D}$. Then u is harmonic on $\mathbf{D}$.

Proof If $w \in \mathbf{D}$, then $w = rz$ for some $r \in [0,1)$ and some $z \in \partial\mathbf{D}$. Thus

$$\begin{aligned} u(w) &= (\mathcal{P}_r f)(z) \\ &= \sum_{n=0}^{\infty} \hat{f}(n)(rz)^n + \sum_{n=1}^{\infty} \hat{f}(-n)(r\bar{z})^n \\ &= \sum_{n=0}^{\infty} \hat{f}(n)w^n + \overline{\sum_{n=1}^{\infty} \overline{\hat{f}(-n)}w^n}. \end{aligned}$$

Every function that has a power series representation on $\mathbf{D}$ is analytic on $\mathbf{D}$. Thus the equation above shows that u is the sum of an analytic function and the complex conjugate of an analytic function. Hence u is harmonic. ∎

11.23 *Poisson integral solves Dirichlet problem on unit disk*

Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is continuous. Define $u\colon \overline{\mathbf{D}} \to \mathbf{C}$ by

$$u(rz) = \begin{cases} (\mathcal{P}_r f)(z) & \text{if } 0 \le r < 1 \text{ and } z \in \partial\mathbf{D}, \\ f(z) & \text{if } r = 1 \text{ and } z \in \partial\mathbf{D}. \end{cases}$$

Then u is continuous on $\overline{\mathbf{D}}$, $u|_{\mathbf{D}}$ is harmonic, and $u|_{\partial\mathbf{D}} = f$.

Proof Suppose $\zeta \in \partial\mathbf{D}$. To prove that u is continuous at ζ, we need to show that if $w \in \overline{\mathbf{D}}$ is close to ζ, then $u(w)$ is close to $u(\zeta)$. Because $u|_{\partial\mathbf{D}} = f$ and f is continuous on $\partial\mathbf{D}$, we do not need to worry about the case where $w \in \partial\mathbf{D}$. Thus assume $w \in \mathbf{D}$. We can write $w = rz$, where $r \in [0,1)$ and $z \in \partial\mathbf{D}$. Now

$$\begin{aligned} |u(\zeta) - u(w)| &= |f(\zeta) - (\mathcal{P}_r f)(z)| \\ &\le |f(\zeta) - f(z)| + |f(z) - (\mathcal{P}_r f)(z)|. \end{aligned}$$

If w is close to ζ, then z is also close to ζ, and hence by the continuity of f the first term in the last line above is small. Also, if w is close to ζ, then r is close to 1, and hence by 11.18 the second term in the last line above is small. Thus if w is close to ζ, then $u(w)$ is close to $u(\zeta)$, as desired.

The function $u|_{\mathbf{D}}$ is harmonic on $\mathbf{D}$ (and hence continuous on $\mathbf{D}$) by 11.22.

The definition of u immediately implies that $u|_{\partial\mathbf{D}} = f$. ∎

Fourier Series of Smooth Functions

The Fourier series of a continuous function on $\partial\mathbf{D}$ need not converge pointwise (see Exercise 11). However, in this subsection we will see that Fourier series behave well for functions that are twice continuously differentiable.

First we need to define what we mean for a function on $\partial\mathbf{D}$ to be differentiable. The formal definition is given below, along with the introduction of the notation $\tilde{f}$ for the transfer of f to $(-\pi, \pi]$ and $f^{[k]}$ for the transfer back to $\partial\mathbf{D}$ of the kth-derivative of $\tilde{f}$.

The idea here is that we transfer a function defined on $\partial\mathbf{D}$ to $(-\pi, \pi]$, take the usual derivative there, then transfer back to $\partial\mathbf{D}$.

11.24 Definition *$\tilde{f}$; k times continuously differentiable; $f^{[k]}$*

Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is a complex-valued function on $\partial\mathbf{D}$ and $k \in \mathbf{Z}^+ \cup \{0\}$.

- Define $\tilde{f}\colon \mathbf{R} \to \mathbf{C}$ by $\tilde{f}(t) = f(e^{it})$.
- f is called k times *continuously differentiable* if $\tilde{f}$ is k times differentiable everywhere on $\mathbf{R}$ and its kth-derivative $\tilde{f}^{(k)}\colon \mathbf{R} \to \mathbf{C}$ is continuous.
- If f is k times continuously differentiable, then $f^{[k]}\colon \partial\mathbf{D} \to \mathbf{C}$ is defined by
$$f^{[k]}(e^{it}) = \tilde{f}^{(k)}(t)$$
for $t \in \mathbf{R}$. Here $\tilde{f}^{(0)}$ is defined to be $\tilde{f}$, which means that $f^{[0]} = f$.

Note that the function $\tilde{f}$ defined above is periodic on $\mathbf{R}$ because $\tilde{f}(t + 2\pi) = \tilde{f}(t)$ for all $t \in \mathbf{R}$. Thus all derivatives of $\tilde{f}$ are also periodic on $\mathbf{R}$.

11.25 Example Suppose $n \in \mathbf{Z}$ and $f\colon \partial\mathbf{D} \to \mathbf{C}$ is defined by $f(z) = z^n$. Then $\tilde{f}\colon \mathbf{R} \to \mathbf{C}$ is defined by $\tilde{f}(t) = e^{int}$.

If $k \in \mathbf{Z}^+$, then $\tilde{f}^{(k)}(t) = i^k n^k e^{int}$. Thus $f^{[k]}(z) = i^k n^k z^n$ for $z \in \partial\mathbf{D}$.

Our next result gives a formula for the Fourier coefficients of a derivative.

11.26 ***Fourier coefficients of differentiable functions***

Suppose $k \in \mathbf{Z}^+$ and $f\colon \partial\mathbf{D} \to \mathbf{C}$ is k times continuously differentiable. Then
$$\bigl(f^{[k]}\bigr)\hat{}\,(n) = i^k n^k \hat{f}(n)$$
for every $n \in \mathbf{Z}$.

Proof First suppose $n = 0$. By the Fundamental Theorem of Calculus, we have
$$(f^{[k]})\hat{}\,(0) = \int_{-\pi}^{\pi} f^{[k]}(e^{it})\,\frac{dt}{2\pi} = \int_{-\pi}^{\pi} \tilde{f}^{(k)}(t)\,\frac{dt}{2\pi} = \tilde{f}^{(k-1)}(t)\Big]_{t=-\pi}^{t=\pi} = 0,$$
which is the desired result for $n = 0$.

Now suppose $n \in \mathbf{Z} \setminus \{0\}$. Then

$$\begin{aligned}(f^{[k]})\hat{\ }(n) &= \int_{-\pi}^{\pi} \tilde{f}^{(k)}(t)e^{-int}\,\frac{dt}{2\pi}\\ &= \frac{1}{2\pi}\tilde{f}^{(k-1)}(t)e^{-int}\Big]_{t=-\pi}^{t=\pi} + in\int_{-\pi}^{\pi}\tilde{f}^{(k-1)}(t)e^{-int}\,\frac{dt}{2\pi}\\ &= in\big(f^{[k-1]}\big)\hat{\ }(n),\end{aligned}$$

where the second equality above follows from integration by parts.

Iterating the equation above now produces the desired result. ■

Now we can prove the beautiful result that a twice continuously differentiable function on $\partial\mathbf{D}$ equals its Fourier series, with uniform convergence of the Fourier series. This conclusion holds with the weaker hypothesis that the function is continuously differentiable, but the proof is easier with the hypothesis used here.

11.27 ***Fourier series of twice continuously differentiable functions converge***

Suppose $f : \partial\mathbf{D} \to \mathbf{C}$ is twice continuously differentiable. Then

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$$

for all $z \in \partial\mathbf{D}$. Furthermore, the partial sums $\sum_{n=-K}^{M} \hat{f}(n)z^n$ converge uniformly on $\partial\mathbf{D}$ to f as $K, M \to \infty$.

Proof If $n \in \mathbf{Z} \setminus \{0\}$, then

11.28 $$|\hat{f}(n)| = \frac{|(f^{[2]})\hat{\ }(n)|}{n^2} \le \frac{\|f^{[2]}\|_1}{n^2},$$

where the equality above follows from 11.26 and the inequality above follows from 11.9(c). Now 11.28 implies that

11.29 $$\sum_{n=-\infty}^{\infty} |\hat{f}(n)z^n| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

for all $z \in \partial\mathbf{D}$. The inequality above implies that $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ converges and that the partial sums converge uniformly on $\partial\mathbf{D}$.

Furthermore, for each $\in \partial\mathbf{D}$ we have

$$f(z) = \lim_{r\uparrow 1} \sum_{n=-\infty}^{\infty} r^{|n|}\hat{f}(n)z^n = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n,$$

where the first equality holds by 11.18 and 11.11, and the second equality holds by the Dominated Convergence Theorem (use counting measure on $\mathbf{Z}$) and 11.29. ■

In 1923 Andrey Kolmogorov (1903–1987) published a proof that there exists a function in $L^1(\partial\mathbf{D})$ whose Fourier series diverges almost everywhere on $\partial\mathbf{D}$. Kolmogorov's result and the result in Exercise 11 probably led most mathematicians to suspect that there exists a continuous function on $\partial\mathbf{D}$ whose Fourier series diverges almost everywhere. However, in 1966 Lennart Carleson (1928–) showed that if $f \in L^2(\partial\mathbf{D})$ (and in particular if f is continuous on $\partial\mathbf{D}$), then the Fourier series of f converges to f almost everywhere.

EXERCISES 11A

1 Prove that $(\overline{f})\hat{}(n) = \overline{\hat{f}(-n)}$ for all $f \in L^1(\partial\mathbf{D})$ and all $n \in \mathbf{Z}$.

2 Suppose $1 \le p \le \infty$ and $n \in \mathbf{Z}$.

(a) Show that the function $f \mapsto \hat{f}(n)$ is a bounded linear functional on $L^p(\partial\mathbf{D})$ with norm 1.

(b) Find all $f \in L^p(\partial\mathbf{D})$ such that $\|f\|_p = 1$ and $|\hat{f}(n)| = 1$.

3 Show that if $0 \le r < 1$ and $t \in \mathbf{R}$, then

$$P_r(e^{it}) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

4 Suppose $f \in \mathcal{L}^1(\partial\mathbf{D})$, $z \in \partial\mathbf{D}$, and f is continuous at z. Prove that

$$\lim_{r\uparrow 1} (\mathcal{P}_r f)(z) = f(z).$$

[*Here $\mathcal{L}^1(\partial\mathbf{D})$ means the complex version of $\mathcal{L}^1(\sigma)$. The result in this exercise differs from 11.18 because here we are assuming continuity only at a single point and we are not even assuming that f is bounded, as compared to 11.18, which assumed continuity at all points of $\partial\mathbf{D}$.*]

5 Suppose $f \in \mathcal{L}^1(\partial\mathbf{D})$, $z \in \partial\mathbf{D}$, $\lim_{t\downarrow 0} f(e^{it}z) = a$, and $\lim_{t\uparrow 0} f(e^{it}z) = b$. Prove that

$$\lim_{r\uparrow 1} (\mathcal{P}_r f)(z) = \frac{a + b}{2}.$$

[*If $a \ne b$, then f is said to have a* jump discontinuity *at z.*]

6 Prove that for each $p \in [1, \infty)$, there exists $f \in L^1(\partial\mathbf{D})$ such that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p = \infty.$$

7 Suppose $\zeta \in \partial\mathbf{D}$. Show that the function

$$w \mapsto \frac{1 - |w|^2}{|1 - \overline{\zeta}w|^2}$$

is harmonic on $\mathbf{C} \setminus \{\zeta\}$ by finding an analytic function on $\mathbf{C} \setminus \{\zeta\}$ whose real part is the function above.

8 Suppose $f\colon \partial\mathbf{D} \to \mathbf{R}$ is the function defined by

$$f(x,y) = x^4 y$$

for $(x,y) \in \mathbf{R}^2$ with $x^2 + y^2 = 1$. Find a polynomial u of two variables x, y such that u is harmonic on $\mathbf{R}^2$ and $u|_{\partial\mathbf{D}} = f$.

[*Of course, $u|_{\mathbf{D}}$ is the Poisson integral of f. However, here you are asked to find an explicit formula for u in closed form, without involving or computing an integral. It may help to think of f as defined by $f(z) = (\operatorname{Re} z)^4 (\operatorname{Im} z)$ for $z \in \partial\mathbf{D}$.*]

9 Find a formula (in closed form, not as an infinite sum) for $\mathcal{P}_r f$, where f is the function in the second bullet point of Example 11.8.

10 Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is three times continuously differentiable. Prove that

$$f^{[1]}(z) = i \sum_{n=-\infty}^{\infty} n\hat{f}(n) z^n$$

for all $z \in \partial\mathbf{D}$.

11 Let $C(\partial\mathbf{D})$ denote the Banach space of continuous function from $\partial\mathbf{D}$ to $\mathbf{C}$, with the supremum norm. For $M \in \mathbf{Z}^+$, define a linear functional $\varphi_M\colon C(\partial\mathbf{D}) \to \mathbf{C}$ by

$$\varphi_M(f) = \sum_{n=-M}^{M} \hat{f}(n).$$

Thus $\varphi_M(f)$ is a partial sum of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$, evaluated at $z = 1$.

(a) Show that

$$\varphi_M(f) = \int_{-\pi}^{\pi} f(e^{it}) \frac{\sin(M + \frac{1}{2})t}{\sin \frac{t}{2}} \frac{dt}{2\pi}$$

for every $f \in C(\partial\mathbf{D})$ and every $M \in \mathbf{Z}^+$.

(b) Show that

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \left| \frac{\sin(M + \frac{1}{2})t}{\sin \frac{t}{2}} \right| \frac{dt}{2\pi} = \infty.$$

(c) Show that $\lim_{M\to\infty} \|\varphi_M\| = \infty$.

(d) Show that there exists $f \in C(\partial\mathbf{D})$ such that $\lim_{M\to\infty} \sum_{n=-M}^{M} \hat{f}(n)$ does not exist (as an element of $\mathbf{C}$).

[*Because the sum in part (d) is a partial sum of the Fourier series evaluated at $z = 1$, part (d) shows that the Fourier series of a continuous function on $\partial\mathbf{D}$ need not converge pointwise on $\partial\mathbf{D}$.*

The family of functions (one for each $M \in \mathbf{Z}^+$) on $\partial\mathbf{D}$ defined by

$$e^{it} \mapsto \frac{\sin(M + \frac{1}{2})t}{\sin \frac{t}{2}}$$

is called the Dirichlet kernel.]

12 Define $f : \partial\mathbf{D} \to \mathbf{R}$ by

$$f(z) = \begin{cases} 1 & \text{if } \operatorname{Im} z > 0, \\ -1 & \text{if } \operatorname{Im} z < 0, \\ 0 & \text{if } \operatorname{Im} z = 0. \end{cases}$$

(a) Show that if $n \in \mathbf{Z}$, then

$$\hat{f}(n) = \begin{cases} -\frac{2i}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

(b) Show that

$$(\mathcal{P}_r f)(z) = \frac{2}{\pi} \arctan \frac{2r \operatorname{Im} z}{1 - r^2}$$

for every $r \in [0, 1)$ and every $z \in \partial\mathbf{D}$.

(c) Verify that $\lim_{r \uparrow 1} (\mathcal{P}_r f)(z) = f(z)$ for every $z \in \partial\mathbf{D}$.

(d) Prove that $\mathcal{P}_r f$ does not converge uniformly to f on $\partial\mathbf{D}$.

11B Fourier Series and L^p of Unit Circle

The last paragraph of the previous section mentioned the result that the Fourier series of a function in $L^2(\partial\mathbf{D})$ converges pointwise to the function almost everywhere. This terrific result had been an open question until 1966. Its proof is not included in this book, partly because the proof is difficult and partly because pointwise convergence has turned out to be less useful than norm convergence.

Thus we begin this section with the easy proof that the Fourier series converges in the norm of $L^2(\partial\mathbf{D})$. The remainder of this section then concentrates on issues connected with norm convergence.

Orthonormal Basis for L^2 of Unit Circle

We already showed that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal family in $L^2(\partial\mathbf{D})$ (see 11.6). Now we show that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$.

11.30 ***orthonormal basis of*** $L^2(\partial\mathbf{D})$

The family $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$.

Proof Suppose $f \in \big(\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}\big)^{\perp}$. Thus $\langle f, z^n\rangle = 0$ for all $n \in \mathbf{Z}$. In other words, $\hat{f}(n) = 0$ for all $n \in \mathbf{Z}$.

Suppose $\varepsilon > 0$. Let $g\colon \partial\mathbf{D} \to \mathbf{C}$ be a twice continuously differentiable function such that $\|f - g\|_2 < \varepsilon$. [To prove the existence of $g \in L^2(\partial\mathbf{D})$ with this property, first approximate f by step functions as in 3.47, but use the L^2-norm instead of the L^1-norm. Then approximate the characteristic function of an interval as in 3.48, but again use the L^2-norm and round the corners of the graph in the proof of 3.48 to get a twice continuously differentiable function.]

Now

$$\begin{aligned}
\|f\|_2 &\le \|f-g\|_2 + \|g\|_2\\
&= \|f-g\|_2 + \Big(\sum_{n\in\mathbf{Z}} |\hat{g}(n)|^2\Big)^{1/2}\\
&= \|f-g\|_2 + \Big(\sum_{n\in\mathbf{Z}} |(g-f)\hat{\ }(n)|^2\Big)^{1/2}\\
&\le \|f-g\|_2 + \|g-f\|_2\\
&< 2\varepsilon,
\end{aligned}$$

where the second line above follows from 11.27, the third line above holds because $\hat{f}(n) = 0$ for all $n \in \mathbf{Z}$, and the fourth line above follows from Bessel's inequality (8.57).

Because the inequality above holds for all $\varepsilon > 0$, we conclude that $f = 0$. We have now shown that $\big(\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}\big)^{\perp} = \{0\}$. Hence $\overline{\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}} = L^2(\partial\mathbf{D})$ by 8.42, which implies that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$. ∎

Now the convergence of the Fourier series of $f \in L^2(\partial\mathbf{D})$ to f follows immediately from standard Hilbert space theory [see 8.63(a)] and the previous result. Thus with no further proof needed, we have the following important result.

11.31 *convergence of Fourier series in the norm of* $L^2(\partial\mathbf{D})$

Suppose $f \in L^2(\partial\mathbf{D})$. Then

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

where the infinite sum converges to f in the norm of $L^2(\partial\mathbf{D})$.

The next example is a spectacular application of Hilbert space theory and the orthonormal basis $\{z^n\}_{n\in\mathbf{Z}}$ of $L^2(\partial\mathbf{D})$. The evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ had been an open question until Euler discovered in 1734 that this infinite sum equals $\frac{\pi^2}{6}$.

Euler's proof, which would not be considered sufficiently rigorous by today's standards, was quite different from the technique used in the example below.

11.32 Example $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$

Define $f \in L^2(\partial\mathbf{D})$ by $f(e^{it}) = t$ for $t \in (-\pi, \pi]$. Then $\hat{f}(0) = \int_{-\pi}^{\pi} t \, \frac{dt}{2\pi} = 0$. For $n \in \mathbf{Z} \setminus \{0\}$, we have

$$\begin{aligned} \hat{f}(n) &= \int_{-\pi}^{\pi} t e^{-int} \, \frac{dt}{2\pi} \\ &= \frac{te^{int}}{-i2\pi n}\Big]_{t=-\pi}^{t=\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-int} \, \frac{dt}{2\pi} \\ &= \frac{(-1)^n i}{n}, \end{aligned}$$

where the second line above follows from integration by parts. The equation above implies that

11.33
$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Also,

11.34
$$\|f\|_2^2 = \int_{-\pi}^{\pi} t^2 \, \frac{dt}{2\pi} = \frac{\pi^2}{3}.$$

Parseval's identity [8.63(c)] implies that the left side of 11.33 equals the left side of 11.34. Setting the right side of 11.33 equal to the right side of 11.34 shows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Convolution on Unit Circle

Recall that

11.35
$$(\mathcal{P}_r f)(z) = \int_{\partial \mathbf{D}} f(w) P_r(z\overline{w})\, d\sigma(w)$$

for $f \in L^1(\partial \mathbf{D})$, $0 \le r < 1$, and $z \in \partial \mathbf{D}$ (see 11.15). The kind of integral formula that appears in the result above is so useful that it gets a special name and notation.

11.36 Definition ***convolution; $f * g$***

Suppose $f, g \in L^1(\partial \mathbf{D})$. The *convolution* of f and g is denoted $f * g$ and is the function defined by

$$(f * g)(z) = \int_{\partial \mathbf{D}} f(w) g(z\overline{w})\, d\sigma(w)$$

for those $z \in \partial \mathbf{D}$ for which the integral above makes sense.

Thus 11.35 states that $\mathcal{P}_r f = f * P_r$. Here $f \in L^1(\partial \mathbf{D})$ and $P_r \in L^\infty(\partial \mathbf{D})$; hence there is no problem with the integral in the definition of $f * P_r$ being defined for all $z \in \partial \mathbf{D}$. See Exercise 11 for an interpretation of convolution when the functions are transferred to the real line.

The definition above of the convolution of two functions allows both functions to be in $L^1(\partial \mathbf{D})$. The product of two functions in $L^1(\partial \mathbf{D})$ is not, in general, in $L^1(\partial \mathbf{D})$. Thus it is not obvious that the convolution of two functions in $L^1(\partial \mathbf{D})$ is defined anywhere. However, the next result shows that all is well.

11.37 ***convolution of two functions in $L^1(\partial \mathbf{D})$ is in $L^1(\partial \mathbf{D})$***

If $f, g \in L^1(\partial \mathbf{D})$, then $(f * g)(z)$ is defined for almost every $z \in \partial \mathbf{D}$. Furthermore, $f * g \in L^1(\partial \mathbf{D})$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

Proof Suppose $f, g \in \mathcal{L}^1(\partial \mathbf{D})$. The function $(w, z) \mapsto f(w) g(z\overline{w})$ is a measurable function on $\partial \mathbf{D} \times \partial \mathbf{D}$, as you are asked to show in Exercise 4. Now Tonelli's Theorem (5.28) and 11.17 imply that

$$\begin{aligned}\int_{\partial \mathbf{D}} \int_{\partial \mathbf{D}} |f(w) g(z\overline{w})|\, d\sigma(w)\, d\sigma(z) &= \int_{\partial \mathbf{D}} |f(w)| \int_{\partial \mathbf{D}} |g(z\overline{w})|\, d\sigma(z)\, d\sigma(w)\\ &= \int_{\partial \mathbf{D}} |f(w)| \|g\|_1\, d\sigma(w)\\ &= \|f\|_1 \|g\|_1.\end{aligned}$$

The equation above implies that $\int_{\partial \mathbf{D}} |f(w) g(z\overline{w})|\, d\sigma(w) < \infty$ for almost every $z \in \partial \mathbf{D}$. Thus $(f * g)(z)$ is defined for almost every $z \in \partial \mathbf{D}$.

The equation above also implies that $\|f * g\|_1 \le \|f\|_1 \|g\|_1$. ■

Soon we will apply convolution results to Poisson integrals. However, first we need to extend the previous result by bounding $\|f * g\|_p$ when $g \in L^p(\partial \mathbf{D})$.

11.38 L^p-norm of a convolution

Suppose $1 \leq p \leq \infty$, $f \in L^1(\partial\mathbf{D})$, and $g \in L^p(\partial\mathbf{D})$. Then

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Proof We use the following result to estimate the norm in $L^p(\partial\mathbf{D})$:

If $F\colon \partial\mathbf{D} \to \mathbf{C}$ is measurable and $1 \leq p \leq \infty$, then

11.39 $$\|F\|_p = \sup\{\int_{\partial\mathbf{D}} |Fh|\, d\sigma : h \in L^{p'}(\partial\mathbf{D}) \text{ and } \|h\|_{p'} = 1\}.$$

Hölder's inequality (7.9) shows that the left side of the equation above is greater than or equal to the right side. The inequality in the other direction almost follows from 7.12, but 7.12 would require the hypothesis that $f \in L^p(\partial\mathbf{D})$ (and we want the equation above to hold even if $\|f\|_p = \infty$). To get around this problem, apply 7.12 to truncations of F and use the Monotone Convergence Theorem (3.11); the details of verifying 11.39 are left to the reader.

Suppose $h \in L^{p'}(\partial\mathbf{D})$ and $\|h\|_{p'} = 1$. Then

$$\begin{aligned}
\int_{\partial\mathbf{D}} |(f * g)(z)h(z)|\, d\sigma(z) &\leq \int_{\partial\mathbf{D}} \Bigl(\int_{\partial\mathbf{D}} |f(w)g(z\overline{w})|\, d\sigma(w)|h(z)|\Bigr)\, d\sigma(z)\\
&= \int_{\partial\mathbf{D}} |f(w)| \int_{\partial\mathbf{D}} |g(z\overline{w})h(z)|\, d\sigma(z)\, d\sigma(w)\\
&\leq \int_{\partial\mathbf{D}} |f(w)|\|g\|_p\|h\|_{p'}\, d\sigma(w)\\
&= \|f\|_1 \|g\|_p, \qquad (11.40)
\end{aligned}$$

where the second line above follows from Tonelli's Theorem (5.28) and the third line follows from Hölder's inequality (7.9) and 11.17. Now 11.39 (with $F = f * g$) and 11.40 imply that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. ■

Order does not matter in convolutions, as we now prove.

11.41 convolution is commutative

Suppose $f, g \in L^1(\partial\mathbf{D})$. Then $f * g = g * f$.

Proof Suppose $z \in \partial\mathbf{D}$ is such that $(f * g)(z)$ is defined. Then

$$(f * g)(z) = \int_{\partial\mathbf{D}} f(w)g(z\overline{w})\, d\sigma(w) = \int_{\partial\mathbf{D}} f(z\overline{\zeta})g(\zeta)\, d\sigma(\zeta) = (g * f)(z),$$

where the second equality follows from making the substitution $\zeta = z\overline{w}$ (which implies that $w = z\overline{\zeta}$); the invariance of the integral under this substitution is explained in connection with 11.17. ■

Now we come to a major result, stating that for $p \in [1, \infty)$, the Poisson integrals of functions in $L^p(\partial\mathbf{D})$ converge in the norm of $L^p(\partial\mathbf{D})$. This result fails for $p = \infty$ [see, for example, Exercise 12(d) in Section 11A].

11.42 *if $f \in L^p(\partial\mathbf{D})$, then $\mathcal{P}_r f$ converges to f in $L^p(\partial\mathbf{D})$*

Suppose $1 \leq p < \infty$ and $f \in L^p(\partial\mathbf{D})$. Then $\lim\limits_{r \uparrow 1} \|f - \mathcal{P}_r f\|_p = 0$.

Proof Suppose $\varepsilon > 0$. Let $g \colon \partial\mathbf{D} \to \mathbf{C}$ be a continuous function on $\partial\mathbf{D}$ such that

$$\|f - g\|_p < \varepsilon.$$

By 11.18, there exists $R \in [0, 1)$ such that

$$\|g - \mathcal{P}_r g\|_\infty < \varepsilon$$

for all $r \in (R, 1)$. If $r \in (R, 1)$, then

$$\begin{aligned}
\|f - \mathcal{P}_r f\|_p &\leq \|f - g\|_p + \|g - \mathcal{P}_r g\|_p + \|\mathcal{P}_r g - \mathcal{P}_r f\|_p \\
&< \varepsilon + \|g - \mathcal{P}_r g\|_\infty + \|\mathcal{P}_r (g - f)\|_p \\
&< 2\varepsilon + \|P_r * (g - f)\|_p \\
&\leq 2\varepsilon + \|P_r\|_1 \, \|g - f\|_p \\
&< 3\varepsilon,
\end{aligned}$$

where the third line above is justified by 11.41, the fourth line above is justified by 11.38, and the last line above is justified by the equation $\|P_r\|_1 = 1$, which follows from 11.16(a) and 11.16(b). The last inequality implies that $\lim\limits_{r \uparrow 1} \|f - \mathcal{P}_r f\|_p = 0$. ■

As a consequence of the result above, we can now prove that functions in $L^1(\partial\mathbf{D})$, and thus functions in $L^p(\partial\mathbf{D})$ for every $p \in [1, \infty]$, are uniquely determined by their Fourier coefficients. Specifically, if $g, h \in L^1(\partial\mathbf{D})$ and $\hat{g}(n) = \hat{h}(n)$ for every $n \in \mathbf{Z}$, then applying the result below to $g - h$ shows that $g = h$.

11.43 *functions are determined by their Fourier coefficients*

Suppose $f \in L^1(\partial\mathbf{D})$ and $\hat{f}(n) = 0$ for every $n \in \mathbf{Z}$. Then $f = 0$.

Proof Because $\mathcal{P}_r f$ is defined in terms of Fourier coefficients (see 11.11), we know that $\mathcal{P}_r f = 0$ for all $r \in [0, 1)$. Because $\mathcal{P}_r f \to f$ in $L^1(\partial\mathbf{D})$ as $r \uparrow 1$ [by 11.42]), this implies that $f = 0$. ■

Our next result shows that multiplication of Fourier coefficients corresponds to convolution of the corresponding functions.

11.44 *Fourier coefficients of a convolution*

Suppose $f, g \in L^1(\partial\mathbf{D})$. Then

$$(f * g)\hat{}(n) = \hat{f}(n)\,\hat{g}(n)$$

for every $n \in \mathbf{Z}$.

Proof First note that if $w \in \partial\mathbf{D}$ and $n \in \mathbf{Z}$, then

11.45 $$\int_{\partial\mathbf{D}} g(z\overline{w})\overline{z^n}\,d\sigma(z) = \int_{\partial\mathbf{D}} g(\zeta)\overline{\zeta^n w^n}\,d\sigma(\zeta) = \overline{w^n}\hat{g}(n),$$

where the first equality comes from the substitution $\zeta = z\overline{w}$ (equivalent to $z = \zeta w$), which is justified by the rotation invariance of σ.

Now

$$\begin{aligned}
(f * g)\hat{}(n) &= \int_{\partial\mathbf{D}} (f * g)(z)\overline{z^n}\,d\sigma(z)\\
&= \int_{\partial\mathbf{D}} \overline{z^n} \int_{\partial\mathbf{D}} f(w)g(z\overline{w})\,d\sigma(w)\,d\sigma(z)\\
&= \int_{\partial\mathbf{D}} f(w) \int_{\partial\mathbf{D}} g(z\overline{w})\overline{z^n}\,d\sigma(z)\,d\sigma(w)\\
&= \int_{\partial\mathbf{D}} f(w)\overline{w^n}\hat{g}(n)\,d\sigma(w)\\
&= \hat{f}(n)\,\hat{g}(n),
\end{aligned}$$

where the interchange of integration order in the third equality is justified by the same steps used in the proof of 11.37 and the fourth equality above is justified by 11.45. ∎

The next result could be proved by appropriate uses of Tonelli's Theorem and Fubini's Theorem. However, the slick proof technique used in the proof below should be useful in dealing with some of the exercises.

11.46 *convolution is associative*

Suppose $f, g, h \in L^1(\partial\mathbf{D})$. Then $(f * g) * h = f * (g * h)$.

Proof Suppose $n \in \mathbf{Z}$. Using 11.44 twice, we have

$$((f * g) * h)\hat{}(n) = (f * g)\hat{}(n)\hat{h}(n) = \hat{f}(n)\hat{g}(n)\hat{h}(n).$$

Similarly,

$$(f * (g * h))\hat{}(n) = \hat{f}(n)(g * h)\hat{}(n) = \hat{f}(n)\hat{g}(n)\hat{h}(n).$$

Hence $(f * g) * h$ and $f * (g * h)$ have the same Fourier coefficients. Because functions in $L^1(\partial\mathbf{D})$ are determined by their Fourier coefficients (see 11.43), this implies that $(f * g) * h = f * (g * h)$. ∎

EXERCISES 11B

1 Show that the family $\{e_k\}_{k\in\mathbf{Z}}$ of trigonometric functions defined by 11.1 is an orthonormal basis of $L^2\big((-\pi,\pi]\big)$.

2 Use the result of Exercise 12(a) in Section 11A to show that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}.$$

3 Use techniques similar to Example 11.32 to evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$.

[*If you feel industrious, you may also want to evaluate* $\sum_{n=1}^{\infty}1/n^6$. *Similar techniques work to evaluate* $\sum_{n=1}^{\infty}1/n^k$ *for each positive even integer k. You can become famous if you figure out how to evaluate* $\sum_{n=1}^{\infty}1/n^3$, *which currently is an open question.*]

4 Suppose $f,g\colon\partial\mathbf{D}\to\mathbf{C}$ are measurable functions. Prove that the function $(w,z)\mapsto f(w)g(z\overline{w})$ is a measurable function from $\partial\mathbf{D}\times\partial\mathbf{D}$ to $\mathbf{C}$.
[*Here the σ-algebra on* $\partial\mathbf{D}\times\partial\mathbf{D}$ *is the usual product σ-algebra as defined in* 5.2.]

5 Where does the proof of 11.42 fail when $p=\infty$?

6 Suppose $f\in L^1(\partial\mathbf{D})$. Prove that f is real valued (almost everywhere) if and only if $\hat{f}(-n)=\overline{\hat{f}(n)}$ for every $n\in\mathbf{Z}$.

7 Suppose $f\in L^1(\partial\mathbf{D})$. Show that $f\in L^2(\partial\mathbf{D})$ if and only if $\displaystyle\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2<\infty$.

8 Suppose $f\in L^2(\partial\mathbf{D})$. Prove that $|f(z)|=1$ for almost every $z\in\partial\mathbf{D}$ if and only if

$$\sum_{k=-\infty}^{\infty}\hat{f}(k)\overline{\hat{f}(k-n)}=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{if } n\neq 0\end{cases}$$

for all $n\in\mathbf{Z}$.

9 For this exercise, for each $r\in[0,1)$ think of $\mathcal{P}_r$ as an operator on $L^2(\partial\mathbf{D})$.

(a) Show that $\mathcal{P}_r$ is a self-adjoint compact operator for each $r\in[0,1)$.

(b) For each $r\in[0,1)$, find all eigenvalues and eigenvectors of $\mathcal{P}_r$.

(c) Prove or disprove: $\lim_{r\uparrow 1}\|I-\mathcal{P}_r\|=0$.

10 Suppose $f\in L^1(\partial\mathbf{D})$. Define $T\colon L^2(\partial\mathbf{D})\to L^2(\partial\mathbf{D})$ by $Tg=f*g$.

(a) Show that T is a compact operator on $L^2(\partial\mathbf{D})$.

(b) Prove that T is injective if and only if $\hat{f}(n)\neq 0$ for every $n\in\mathbf{Z}$.

(c) Find a formula for T^*.

(d) Prove: T is self-adjoint if and only if all Fourier coefficients of f are real.

(e) Show that T is a normal operator.

11 Show that if $f, g \in L^1(\partial\mathbf{D})$ then

$$(f * g)\tilde{}(t) = \int_{-\pi}^{\pi} \tilde{f}(x)\tilde{g}(t-x)\,dx,$$

for those $t \in \mathbf{R}$ such that $(f * g)(e^{it})$ makes sense; here $(f * g)\tilde{}$, $\tilde{f}$, and $\tilde{g}$ denote the transfers to the real line as defined in 11.24.

12 Suppose $1 \leq p \leq \infty$. Prove that if $f \in L^p(\partial\mathbf{D})$ and $g \in L^{p'}(\partial\mathbf{D})$, then $f * g$ is a continuous function on $\partial\mathbf{D}$.

13 Suppose $g \in L^1(\partial\mathbf{D})$ is such that $\hat{g}(n) \neq 0$ for infinitely many $n \in \mathbf{Z}$. Prove that if $f \in L^1(\partial\mathbf{D})$ and $f * g = g$, then $f = 0$.

14 Show that there exists a two-sided sequence $\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ such that $\lim\limits_{n\to\pm\infty} b_n = 0$ but there does not exist $f \in L^1(\partial\mathbf{D})$ with $\hat{f}(n) = b_n$ for all $n \in \mathbf{Z}$.

15 Prove that if $f, g \in L^2(\partial\mathbf{D})$, then

$$(fg)\hat{}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$$

for every $n \in \mathbf{Z}$.

16 Suppose $f \in L^1(\partial\mathbf{D})$. Prove that $\mathcal{P}_r(\mathcal{P}_s f) = \mathcal{P}_{rs} f$ for all $r, s \in [0,1)$.

17 Suppose $p \in [1, \infty]$ and $f \in L^p(\partial\mathbf{D})$. Prove that if $0 \leq r < s < 1$, then

$$\|\mathcal{P}_r f\|_p \leq \|\mathcal{P}_s f\|_p.$$

18 Prove Wirtinger's inequality: If $f\colon \mathbf{R} \to \mathbf{R}$ is a continuously differentiable 2π-periodic function and $\int_{-\pi}^{\pi} f(t)\,dt = 0$, then

$$\int_{-\pi}^{\pi} \bigl(f(t)\bigr)^2\,dt \leq \int_{-\pi}^{\pi} \bigl(f'(t)\bigr)^2\,dt,$$

with equality if and only if $f(t) = a\sin(t) + b\cos(t)$ for some constants a, b.

11C Fourier Transform

Fourier Transform on $L^1(\mathbf{R})$

We now switch from consideration of functions defined on the unit circle $\partial\mathbf{D}$ to consideration of functions defined on the real line $\mathbf{R}$. Instead of dealing with Fourier coefficients and Fourier series, we now deal with Fourier transforms.

Recall that $\int_{-\infty}^{\infty} f(x)\,dx$ means $\int_{\mathbf{R}} f\,d\lambda$, where λ denotes Lebesgue measure on $\mathbf{R}$, and similarly if a dummy variable other than x is used (see 3.39). Similarly, $L^p(\mathbf{R})$ means $L^p(\lambda)$ (the version that allows the functions to be complex valued). Thus in this section, $\|f\|_p = \left(\int_{-\infty}^{\infty}|f(x)|^p\,dx\right)^{1/p}$ for $1 \le p < \infty$.

11.47 Definition ***Fourier transform***

For $f \in L^1(\mathbf{R})$, the *Fourier transform* of f is the function $\hat{f}\colon \mathbf{R} \to \mathbf{C}$ defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx.$$

We use the same notation $\hat{f}$ for the Fourier transform as we did for Fourier coefficients. The analogies that we will see between the two concepts makes using the same notation reasonable. The context should make it clear whether this notation refers to Fourier transforms (when we are working with functions defined on $\mathbf{R}$) or whether the notation refers to Fourier coefficients (when we are working with functions defined on $\partial\mathbf{D}$).

The factor 2π that appears in the exponent in the definition above of the Fourier transform is a normalization factor. Without this normalization, we would lose the beautiful result that $\|\hat{f}\|_2 = \|f\|_2$ (see 11.82). Another possible normalization, which is used by some books, is to define the Fourier transform of f at t to be

$$\int_{-\infty}^{\infty} f(x)e^{-itx}\,\frac{dx}{\sqrt{2\pi}}.$$

There is no right or wrong way to do the normalization—pesky π's will pop up somewhere regardless of the normalization or lack of normalization. However, the choice made in 11.47 seems to cause fewer problems than other choices.

11.48 Example *Fourier transforms*

(a) Suppose $b \le c$. If $t \in \mathbf{R}$, then

$$\begin{aligned}(\chi_{[b,c]})\hat{\ }(t) &= \int_b^c e^{-2\pi itx}\,dx \\ &= \begin{cases} \dfrac{i\left(e^{-2\pi ict} - e^{-2\pi ibt}\right)}{2\pi t} & \text{if } t \neq 0, \\ c - b & \text{if } t = 0. \end{cases}\end{aligned}$$

(b) Suppose $f(x) = e^{-2\pi|x|}$ for $x \in \mathbf{R}$. If $t \in \mathbf{R}$, then

$$\begin{aligned}\hat{f}(t) &= \int_{-\infty}^{\infty} e^{-2\pi|x|} e^{-2\pi i t x}\,dx \\ &= \int_{-\infty}^{0} e^{2\pi x} e^{-2\pi i t x}\,dx + \int_{0}^{\infty} e^{-2\pi x} e^{-2\pi i t x}\,dx \\ &= \frac{1}{2\pi(1-it)} + \frac{1}{2\pi(1+it)} \\ &= \frac{1}{\pi(t^2+1)}.\end{aligned}$$

Recall that the Riemann–Lebesgue Lemma on the unit circle $\partial\mathbf{D}$ states that if $f \in L^1(\partial\mathbf{D})$, then $\lim_{n\to\pm\infty} \hat{f}(n) = 0$ (see 11.10). Now we come to the analogous result in the context of the real line.

11.49 *Riemann–Lebesgue Lemma*

Suppose $f \in L^1(\mathbf{R})$. Then $\hat{f}$ is uniformly continuous on $\mathbf{R}$. Furthermore,

$$\|\hat{f}\|_\infty \le \|f\|_1 \quad \text{and} \quad \lim_{t\to\pm\infty} \hat{f}(t) = 0.$$

Proof Because $|e^{-2\pi i t x}| = 1$ for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}$, the definition of the Fourier transform implies that if $t \in \mathbf{R}$ then

$$|\hat{f}(t)| \le \int_{-\infty}^{\infty} |f(x)|\,dx = \|f\|_1.$$

Thus $\|\hat{f}\|_\infty \le \|f\|_1$.

If f is the characteristic function of a bounded interval, then the formula in Example 11.48(a) shows that $\hat{f}$ is uniformly continuous on $\mathbf{R}$ and $\lim_{t\to\pm\infty} \hat{f}(t) = 0$. Thus the same result holds for finite linear combinations of such functions. Such finite linear combinations are called step functions (see 3.46).

Now consider arbitrary $f \in L^1(\mathbf{R})$. There exists a sequence $f_1, f_2, \ldots$ of step functions in $L^1(\mathbf{R})$ such that $\lim_{k\to\infty} \|f - f_k\|_1 = 0$ (by 3.47). Thus

$$\lim_{k\to\infty} \|\hat{f} - \hat{f}_k\|_\infty = 0.$$

In other words, the sequence $\hat{f}_1, \hat{f}_2, \ldots$ converges uniformly on $\mathbf{R}$ to $\hat{f}$. Because the uniform limit of uniformly continuous functions is uniformly continuous, we can conclude that $\hat{f}$ is uniformly continuous on $\mathbf{R}$. Furthermore, the uniform limit of functions on $\mathbf{R}$ each of which has limit 0 at $\pm\infty$ also has limit 0 at $\pm\infty$, completing the proof. ∎

The next result gives a condition that forces the Fourier transform of a function to be continuously differentiable. This result also gives a formula for the derivative of the Fourier transform. See Exercise 8 for a formula for the nth derivative.

11.50 *derivative of a Fourier transform*

Suppose $f \in L^1(\mathbf{R})$. Define $g \colon \mathbf{R} \to \mathbf{C}$ by $g(x) = xf(x)$. If $g \in L^1(\mathbf{R})$, then $\hat{f}$ is a continuously differentiable function on $\mathbf{R}$ and

$$(\hat{f})'(t) = -2\pi i \hat{g}(t)$$

for all $t \in \mathbf{R}$.

Proof Fix $t \in \mathbf{R}$. Then

$$\begin{aligned}
\lim_{s \to 0} \frac{\hat{f}(t+s) - \hat{f}(t)}{s} &= \lim_{s \to 0} \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \Big(\frac{e^{-2\pi i s x} - 1}{s} \Big) \, dx \\
&= \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \Big(\lim_{s \to 0} \frac{e^{-2\pi i s x} - 1}{s} \Big) \, dx \\
&= -2\pi i \int_{-\infty}^{\infty} x f(x) e^{-2\pi i t x} \, dx \\
&= -2\pi i \hat{g}(t),
\end{aligned}$$

where the second equality is justified by using the inequality $|e^{i\theta} - 1| \leq \theta$ (valid for all $\theta \in \mathbf{R}$, as the reader should verify) to show that $|(e^{-2\pi i s x} - 1)/s| \leq 2\pi |x|$ for all $s \in \mathbf{R} \setminus \{0\}$ and all $x \in \mathbf{R}$; the hypothesis that $xf(x) \in L^1(\mathbf{R})$ and the Dominated Convergence Theorem (3.31) then allow for the interchange of the limit and the integral that is used in the second equality above.

The equation above shows that $\hat{f}$ is differentiable and that $(\hat{f})'(t) = -2\pi i \hat{g}(t)$ for all $t \in \mathbf{R}$. Because $\hat{g}$ is continuous on $\mathbf{R}$ (by 11.49), we can also conclude that $\hat{f}$ is continuously differentiable. ∎

11.51 Example $e^{-\pi x^2}$ *equals its Fourier transform*

Suppose $f \in L^1(\mathbf{R})$ is defined by $f(x) = e^{-\pi x^2}$. Then the function $g \colon \mathbf{R} \to \mathbf{C}$ defined by $g(x) = xf(x) = xe^{-\pi x^2}$ is in $L^1(\mathbf{R})$. Hence 11.50 implies that if $t \in \mathbf{R}$ then

$$\begin{aligned}
(\hat{f})'(t) &= -2\pi i \int_{-\infty}^{\infty} x e^{-\pi x^2} e^{-2\pi i t x} \, dx \\
&= \big(i e^{-\pi x^2} e^{-2\pi i t x} \big) \Big]_{x=-\infty}^{x=\infty} - 2\pi t \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i t x} \, dx
\end{aligned}$$

11.52 $$= -2\pi t \hat{f}(t),$$

where the second equality follows from integration by parts (if you are nervous about doing an integration by parts from $-\infty$ to ∞, change each integral to be the limit as $M \to \infty$ of the integral from $-M$ to M).

Note that $f'(t) = -2\pi t e^{-\pi t^2} = -2\pi t f(t)$. Combining this equation with 11.52 shows that

$$\Big(\frac{\hat{f}}{f}\Big)'(t) = \frac{f(t)(\hat{f})'(t) - f'(t)\hat{f}(t)}{\big(f(t)\big)^2} = -2\pi t \frac{f(t)\hat{f}(t) - f(t)\hat{f}(t)}{\big(f(t)\big)^2} = 0$$

for all $t \in \mathbf{R}$. Thus $\hat{f}/f$ is a constant function. In other words, there exists $c \in \mathbf{C}$ such that $\hat{f} = cf$. To evaluate c, note that

11.53
$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1 = f(0),$$

where the integral above is evaluated by writing its square as the integral times the same integral but using y instead of x for the dummy variable and then converting to polar coordinates ($dx\,dy = r\,dr\,d\theta$).

Clearly 11.53 implies that $c = 1$. Thus $\hat{f} = f$.

The next result gives a formula for the Fourier transform of a derivative. See Exercise 9 for a formula for the Fourier transform of the nth derivative.

11.54 *Fourier transform of a derivative*

Suppose $f \in L^1(\mathbf{R})$ is a continuously differentiable function and $f' \in L^1(\mathbf{R})$. If $t \in \mathbf{R}$, then

$$(f')\hat{\ }(t) = 2\pi i t \hat{f}(t).$$

Proof Suppose $\varepsilon > 0$. Because f and f' are in $L^1(\mathbf{R})$, there exists $a \in \mathbf{R}$ such that

$$\int_a^{\infty} |f'(x)|\,dx < \varepsilon \quad \text{and} \quad |f(a)| < \varepsilon.$$

Now if $b > a$ then

$$|f(b)| = \Big|\int_a^b f'(x)\,dx + f(a)\Big| \le \int_a^{\infty} |f'(x)|\,dx + |f(a)| < 2\varepsilon.$$

Hence $\lim_{x\to\infty} f(x) = 0$. Similarly, $\lim_{x\to-\infty} f(x) = 0$.

If $t \in \mathbf{R}$, then

$$\begin{aligned}(f')\hat{\ }(t) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi i t x}\,dx \\ &= f(x)e^{-2\pi i t x}\Big]_{x=-\infty}^{x=\infty} + 2\pi i t \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx \\ &= 2\pi i t \hat{f}(t),\end{aligned}$$

where the second equality comes from integration by parts and the third equality holds because we showed in the paragraph above that $\lim_{x\to\pm\infty} f(x) = 0$. ■

The next result gives formulas for the Fourier transforms of some algebraic transformations of a function. Proofs of these formulas are left to the reader.

11.55 *Fourier transforms of translations, rotations, and dilations*

Suppose $f \in L^1(\mathbf{R})$, $b \in \mathbf{R}$, and $t \in \mathbf{R}$.

(a) If $g(x) = f(x-b)$ for all $x \in \mathbf{R}$, then $\hat{g}(t) = e^{-2\pi ibt}\hat{f}(t)$.

(b) If $g(x) = e^{2\pi ibx}f(x)$ for all $x \in \mathbf{R}$, then $\hat{g}(t) = \hat{f}(t-b)$.

(c) If $b \neq 0$ and $g(x) = f(bx)$ for all $x \in \mathbf{R}$, then $\hat{g}(t) = \frac{1}{|b|}\hat{f}\big(\frac{t}{b}\big)$.

11.56 Example *Fourier transform of a rotation of an exponential function*

Suppose $y > 0$, $x \in \mathbf{R}$, and $h(t) = e^{-2\pi y|t|}e^{2\pi ixt}$. To find the Fourier transform of h, first consider the function g defined by $g(t) = e^{-2\pi y|t|}$. By 11.48(b) and 11.55(c), we have

11.57
$$\hat{g}(t) = \frac{1}{y}\frac{1}{\pi\Big(\big(\frac{t}{y}\big)^2 + 1\Big)} = \frac{1}{\pi}\frac{y}{t^2 + y^2}.$$

Now 11.55(b) implies that

11.58
$$\hat{h}(t) = \frac{1}{\pi}\frac{y}{(t-x)^2 + y^2};$$

note that x is a constant in the definition of h, which has t as the variable, but x is the variable in 11.55(b)—this slightly awkward permutation of variables is done in this example to make a later reference to 11.58 come out cleaner.

The next result will be immensely useful later in this section.

11.59 *integral of a function times a Fourier transform*

Suppose $f, g \in L^1(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} \hat{f}(t)g(t)\,dt = \int_{-\infty}^{\infty} f(t)\hat{g}(t)\,dt.$$

Proof Both integrals in the equation above make sense because $f, g \in L^1(\mathbf{R})$ and $\hat{f}, \hat{g} \in L^\infty(\mathbf{R})$ (by 11.49). Using the definition of the Fourier transform, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(t)g(t)\,dt &= \int_{-\infty}^{\infty} g(t)\int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx\,dt\\ &= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(t)e^{-2\pi itx}\,dt\,dx\\ &= \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx,\end{aligned}$$

where Tonelli's Theorem and Fubini's Theorem justify the second equality. Changing the dummy variable x to t in the last expression gives the desired result. ∎

Convolution on $\mathbf{R}$

Our next big goal is to prove the Fourier Inversion Formula. This remarkable formula, discovered by Fourier, states that if $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$, then

11.60
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i x t}\, dt$$

for almost every $x \in \mathbf{R}$. We will eventually prove this result (see 11.76), but first we need to develop some tools that will be used in the proof. To motivate these tools, we look at the right side of the equation above for fixed $x \in \mathbf{R}$ and see what we would need to prove that it equals $f(x)$.

To get from the right side of 11.60 to an expression involving f rather than $\hat{f}$, we should be tempted to use 11.59. However, we cannot use 11.59 because the function $t \mapsto e^{2\pi i x t}$ is not in $L^1(\mathbf{R})$, which is a hypothesis needed for 11.59. Thus we throw in a convenient convergence factor, fixing $y > 0$ and considering the integral

11.61
$$\int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi y|t|} e^{2\pi i x t}\, dt.$$

The convergence factor above is a good choice because for fixed $y > 0$ the function $t \mapsto e^{-2\pi y|t|}$ is in $L^1(\mathbf{R})$, and $\lim_{y \downarrow 0} e^{-2\pi y|t|} = 1$ for every $t \in \mathbf{R}$ (which means that 11.61 may be a good approximation to 11.60 for y close to 0).

Now let's be rigorous. Suppose $f \in L^1(\mathbf{R})$. Fix $y > 0$ and $x \in \mathbf{R}$. Define $h \colon \mathbf{R} \to \mathbf{C}$ by $h(t) = e^{-2\pi y|t|} e^{2\pi i x t}$. Then $h \in L^1(\mathbf{R})$ and

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi y|t|} e^{2\pi i x t}\, dt &= \int_{-\infty}^{\infty} \hat{f}(t) h(t)\, dt \\
&= \int_{-\infty}^{\infty} f(t) \hat{h}(t)\, dt \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2}\, dt, \qquad (11.62)
\end{aligned}$$

where the second equality comes from 11.59 and the third equality comes from 11.58. We will come back to the specific formula in 11.62 later, but for now we use 11.62 as motivation for study of expressions of the form $\int_{-\infty}^{\infty} f(t) g(x-t)\, dt$. Thus we have been led to the following definition.

11.63 Definition ***convolution; $f * g$***

Suppose $f, g \colon \mathbf{R} \to \mathbf{C}$ are measurable functions. The *convolution* of f and g is denoted $f * g$ and is the function defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

for those $x \in \mathbf{R}$ for which the integral above makes sense.

Here we are using the same terminology and notation as was used for the convolution of functions on the unit circle. Recall that if $F, G \in L^1(\partial\mathbf{D})$, then

$$(F * G)(e^{i\theta}) = \int_{-\pi}^{\pi} F(e^{is})G(e^{i(\theta-s)})\,\frac{ds}{2\pi}$$

for $\theta \in \mathbf{R}$ (see 11.36). The context should always indicate whether $f * g$ denotes convolution on the unit circle or convolution on the real line. The formal similarities between the two notions of convolution make many of the proofs transfer in either direction from one context to the other.

If $f, g \in L^1(\mathbf{R})$, then $f * g$ is defined for almost every $x \in \mathbf{R}$, and furthermore $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ (as you should verify by translating the proof of 11.37 to the context of $\mathbf{R}$).

If $p \in (1, \infty]$, then neither $L^1(\mathbf{R})$ nor $L^p(\mathbf{R})$ is a subset of the other [unlike the inclusion $L^p(\partial\mathbf{D}) \subset L^1(\partial\mathbf{D})$]. Thus we do not yet know that $f * g$ makes sense for $f \in L^1(\mathbf{R})$ and $g \in L^p(\mathbf{R})$. However, the next result shows that all is well.

If $1 \leq p \leq \infty$, $f \in L^p(\mathbf{R})$, and $g \in L^{p'}(\mathbf{R})$, then Hölder's inequality (7.9) and the translation invariance of Lebesgue measure imply $(f * g)(x)$ is defined for all $x \in \mathbf{R}$ and $\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}$ (more is true; with these hypothesis, $f * g$ is a uniformly continuous function on $\mathbf{R}$, as you are asked to show in Exercise 10).

11.64 L^p-norm of a convolution

Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbf{R})$, and $g \in L^p(\mathbf{R})$. Then $(f * g)(x)$ is defined for almost every $x \in \mathbf{R}$. Furthermore,

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Proof First consider the case where $f(x) \geq 0$ and $g(x) \geq 0$ for almost every $x \in \mathbf{R}$. Thus $(f * g)(x)$ is defined for each $x \in \mathbf{R}$, although its value might equal ∞. Apply the proof of 11.38 to the context of $\mathbf{R}$, concluding that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ [which implies that $(f * g)(x) < \infty$ for almost every $x \in \mathbf{R}$].

Now consider arbitrary $f \in L^1(\mathbf{R})$, and $g \in L^p(\mathbf{R})$. Apply the case of the previous paragraph to $|f|$ and $|g|$ to get the desired conclusions. ■

The next proof, as is the case for several other proofs in this section, asks the reader to transfer the proof of the analogous result from the context of the unit circle to the context of the real line. This should require only minor adjustments of a proof from one of the two previous sections. The best way to learn this material is to write out for yourself the required proof in the context of the real line.

11.65 convolution is commutative

Suppose $f, g\colon \mathbf{R} \to \mathbf{C}$ are measurable functions and $x \in \mathbf{R}$ is such that $(f * g)(x)$ is defined. Then $(f * g)(x) = (g * f)(x)$.

Proof Adjust the proof of 11.41 to the context of $\mathbf{R}$. ■

Our next result shows that multiplication of Fourier transforms corresponds to convolution of the corresponding functions.

11.66 *Fourier transform of a convolution*

Suppose $f, g \in L^1(\mathbf{R})$. Then

$$(f * g)\hat{\ }(t) = \hat{f}(t)\, \hat{g}(t)$$

for every $t \in \mathbf{R}$.

Proof Adjust the proof of 11.44 to the context of $\mathbf{R}$. ∎

Poisson Kernel on Upper Half-Plane

As usual, we identify $\mathbf{R}^2$ with $\mathbf{C}$, as illustrated in the following definition. We will see that the upper half-plane plays a role in the context of $\mathbf{R}$ similar to the role that the open unit disk plays in the context of $\partial\mathbf{D}$.

11.67 Definition $\mathbf{H}$*; upper half-plane*

- $\mathbf{H}$ denotes the open upper half-plane in $\mathbf{R}^2$:

$$\mathbf{H} = \{(x, y) \in \mathbf{R}^2 : y > 0\} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}.$$

- $\partial\mathbf{H}$ is identified with the real line:

$$\partial\mathbf{H} = \{(x, y) \in \mathbf{R}^2 : y = 0\} = \{z \in \mathbf{C} : \operatorname{Im} z = 0\} = \mathbf{R}.$$

Recall that we defined a family of functions on $\partial\mathbf{D}$ called the Poisson kernel on $\mathbf{D}$ (see 11.14, where the family is called the Poisson kernel on $\mathbf{D}$ because $0 \le r < 1$ and $\zeta \in \partial\mathbf{D}$ implies $r\zeta \in \mathbf{D}$). Now we are ready to define a family of functions on $\mathbf{R}$ that is called the Poisson kernel on $\mathbf{H}$ [because $x \in \mathbf{R}$ and $y > 0$ implies $(x, y) \in \mathbf{H}$].

The following definition is motivated by 11.62. The notation P_r for the Poisson kernel on $\mathbf{D}$ and P_y for the Poisson kernel on $\mathbf{H}$ is potentially ambiguous (what is $P_{1/2}$?), but the intended meaning should always be clear from the context.

11.68 Definition P_y*; Poisson kernel*

- For $y > 0$, define $P_y \colon \mathbf{R} \to (0, \infty)$ by

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.$$

- The family of functions $\{P_y\}_{y>0}$ is called the *Poisson kernel* on $\mathbf{H}$.

The properties of the Poisson kernel on $\mathbf{H}$ listed in the result below should be compared to the corresponding properties (see 11.16) of the Poisson kernel on $\mathbf{D}$.

11.69 *properties of* P_y

(a) $P_y(x) > 0$ for all $y > 0$ and all $x \in \mathbf{R}$.

(b) $\displaystyle\int_{-\infty}^{\infty} P_y(x)\,dx = 1$ for each $y > 0$.

(c) $\displaystyle\lim_{y\downarrow 0} \int_{\{x\in\mathbf{R}:|x|\geq\delta\}} P_y(x)\,dx = 0$ for each $\delta > 0$.

Proof Part (a) follows immediately from the definition of $P_y(x)$ given in 11.68.

Parts (b) and (c) follow from explicitly evaluating the integrals, using the result that for each $y > 0$, an anti-derivative of $P_y(x)$ (as a function of x) is $\frac{1}{\pi}\arctan\frac{x}{y}$. ■

If $p \in [1, \infty]$ and $f \in L^p(\mathbf{R})$ and $y > 0$, then $f * P_y$ makes sense because $P_y \in L^{p'}(\mathbf{R})$. Thus the following definition makes sense.

11.70 Definition $\mathcal{P}_y f$

For $f \in L^p(\mathbf{R})$ for some $p \in [1, \infty]$ and for $y > 0$, define $\mathcal{P}_y f : \mathbf{R} \to \mathbf{C}$ by

$$(\mathcal{P}_y f)(x) = \int_{-\infty}^{\infty} f(t) P_y(x-t)\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2 + y^2}\,dt$$

for $x \in \mathbf{R}$. In other words, $\mathcal{P}_y f = f * P_y$.

The next result is analogous to 11.18, except that now we need to include in the hypothesis that our function is uniformly continuous and bounded (those conditions follow automatically from continuity in the context of the unit circle).

For the proof of the result below, you should use the properties in 11.69 instead of the corresponding properties in 11.16.

When Napoleon appointed Fourier to an administrative position in 1806, Siméon Poisson (1781–1840) was appointed to the professor position at École Polytechnique vacated by Fourier. Poisson published over 300 mathematical papers during his lifetime.

11.71 *if f is uniformly continuous and bounded, then* $\lim_{y\downarrow 0}\|f - \mathcal{P}_y f\|_\infty = 0$

Suppose $f : \mathbf{R} \to \mathbf{C}$ is uniformly continuous and bounded. Then $\mathcal{P}_y f$ converges uniformly to f on $\mathbf{R}$ as $y \downarrow 0$.

Proof Adjust the proof of 11.18 to the context of $\mathbf{R}$. ■

The function u defined in the result below is called the *Poisson integral* of f on $\mathbf{H}$.

11.72 *Poisson integral is harmonic*

Suppose $f \in L^p(\mathbf{R})$ for some $p \in [1, \infty]$. Define $u \colon \mathbf{H} \to \mathbf{C}$ by

$$u(x,y) = (\mathcal{P}_y f)(x)$$

for $x \in \mathbf{R}$ and $y > 0$. Then u is harmonic on $\mathbf{H}$.

Proof First we consider the case where f is real valued. For $x \in \mathbf{R}$ and $y > 0$, let $z = x + iy$. Then

$$\frac{y}{(x-t)^2 + y^2} = -\operatorname{Im} \frac{1}{z-t}$$

for $t \in \mathbf{R}$. Thus

$$u(x,y) = -\operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1}{z-t}\, dt.$$

The function $z \mapsto -\int_{-\infty}^{\infty} f(t) \frac{1}{z-t}\, dt$ is analytic on $\mathbf{H}$; its derivative is the function $z \mapsto \int_{-\infty}^{\infty} f(t) \frac{1}{(z-t)^2}\, dt$ (justification for this statement is in the next paragraph). In other words, we can differentiate (with respect to z) under the integral sign in the expression above. Because u is the imaginary part of an analytic function, u is harmonic on $\mathbf{H}$, as desired.

To justify the differentiation under the integral sign, fix $z \in \mathbf{H}$ and define a function $g \colon \mathbf{H} \to \mathbf{C}$ by $g(z) = -\int_{-\infty}^{\infty} f(t) \frac{1}{z-t}\, dt$. Then

$$\frac{g(z) - g(w)}{z - w} - \int_{-\infty}^{\infty} f(t) \frac{1}{(z-t)^2}\, dt = \int_{-\infty}^{\infty} f(t) \frac{z - w}{(z-t)^2 (w-t)}\, dt.$$

As $w \to z$, the function $t \mapsto \frac{z-w}{(z-t)^2(w-t)}$ goes to 0 in the norm of $L^{p'}(\mathbf{R})$. Thus Hölder's inequality (7.9) and the equation above imply that $g'(z)$ exists and that $g'(z) = \int_{-\infty}^{\infty} f(t) \frac{1}{(z-t)^2}\, dt$, as desired. ■

We have now solved the Dirichlet problem on the half-space for uniformly continuous, bounded functions on $\mathbf{R}$ (see 11.21 for the statement of the Dirichlet problem).

11.73 *Poisson integral solves Dirichlet problem on half-plane*

Suppose $f \colon \mathbf{R} \to \mathbf{C}$ is uniformly continuous and bounded. Define $u \colon \overline{\mathbf{H}} \to \mathbf{C}$ by

$$u(x,y) = \begin{cases} (\mathcal{P}_y f)(x) & \text{if } x \in \mathbf{R} \text{ and } y > 0, \\ f(x) & \text{if } x \in \mathbf{R} \text{ and } y = 0. \end{cases}$$

Then u is continuous on $\overline{\mathbf{H}}$, $u|_{\mathbf{H}}$ is harmonic, and $u|_{\mathbf{R}} = f$.

Proof Adjust the proof of 11.23 to the context of $\mathbf{R}$; now you will need to use 11.71 and 11.72 instead of the corresponding results for the unit circle. ■

The next result, which states that the Poisson integrals of functions in $L^p(\mathbf{R})$ converge in the norm of $L^p(\mathbf{R})$, will be a major tool in proving the Fourier Inversion Formula and other results later in this section.

Poisson and Fourier are two of the 72 mathematicians/scientists whose names are prominently inscribed on the Eiffel Tower in Paris.

For the result below, the proof of the corresponding result on the unit circle (11.42) does not transfer to the context of $\mathbf{R}$ (because the inequality $\|\cdot\|_p \leq \|\cdot\|_\infty$ fails in the context of $\mathbf{R}$).

11.74 *if $f \in L^p(\mathbf{R})$, then $\mathcal{P}_y f$ converges to f in $L^p(\mathbf{R})$*

Suppose $1 \leq p < \infty$ and $f \in L^p(\mathbf{R})$. Then $\lim_{y \downarrow 0} \|f - \mathcal{P}_y f\|_p = 0$.

Proof If $y > 0$ and $x \in \mathbf{R}$, then

$$|f(x) - (\mathcal{P}_y f)(x)| = \left| f(x) - \int_{-\infty}^{\infty} f(x-t) P_y(t)\, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} \bigl(f(x) - f(x-t)\bigr) P_y(t)\, dt \right|$$

11.75
$$\leq \left(\int_{-\infty}^{\infty} |f(x) - f(x-t)|^p P_y(t)\, dt \right)^{1/p},$$

where the inequality comes from applying 7.10 to the measure $P_y\, dt$ (note that the measure of $\mathbf{R}$ with respect to this measure is 1).

Define $h\colon \mathbf{R} \to [0, \infty)$ by

$$h(t) = \int_{-\infty}^{\infty} |f(x) - f(x-t)|^p\, dx.$$

Then h is a bounded function that is uniformly continuous on $\mathbf{R}$ [by Exercise 23(a) in Section 7A]. Furthermore, $h(0) = 0$.

Raising both sides of 11.75 to the pth power and then integrating over $\mathbf{R}$ with respect to x, we have

$$\|f - \mathcal{P}_y f\|_p^p \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x) - f(x-t)|^p P_y(t)\, dt\, dx$$

$$= \int_{-\infty}^{\infty} P_y(t) \int_{-\infty}^{\infty} |f(x) - f(x-t)|^p\, dx\, dt$$

$$= \int_{-\infty}^{\infty} P_y(-t) h(t)\, dt$$

$$= (\mathcal{P}_y h)(0).$$

Now 11.71 implies that $\lim_{y \downarrow 0} (\mathcal{P}_y h)(0) = h(0) = 0$. Hence the last inequality above implies that $\lim_{y \downarrow 0} \|f - \mathcal{P}_y f\|_p = 0$. ■

Fourier Inversion Formula

Now we can prove the remarkable Fourier Inversion Formula.

11.76 *Fourier Inversion Formula*

Suppose $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$. Then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi ixt} \, dt$$

for almost every $x \in \mathbf{R}$. In other words,

$$f(x) = (\hat{f})\hat{\ }(-x)$$

for almost every $x \in \mathbf{R}$.

Proof Equation 11.62 states that

11.77
$$\int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi y|t|} e^{2\pi ixt} \, dt = (\mathcal{P}_y f)(x)$$

for every $x \in \mathbf{R}$ and every $y > 0$.

Because $\hat{f} \in L^1(\mathbf{R})$, the Dominated Convergence Theorem (3.31) implies that for every $x \in \mathbf{R}$, the left side of 11.77 has limit $(\hat{f})\hat{\ }(-x)$ as $y \downarrow 0$.

Because $f \in L^1(\mathbf{R})$, 11.74 implies that $\lim_{y \downarrow 0} \|f - \mathcal{P}_y f\|_1 = 0$. Now 7.23 implies that there is a sequence of positive numbers $y_1, y_2, \ldots$ such that $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} (\mathcal{P}_{y_n} f)(x) = f(x)$ for almost every $x \in \mathbf{R}$.

Combining the results in the two previous paragraphs and equation 11.77 shows that $f(x) = (\hat{f})\hat{\ }(-x)$ for almost every $x \in \mathbf{R}$. ∎

The Fourier transform of a function in $L^1(\mathbf{R})$ is a uniformly continuous function on $\mathbf{R}$ (by 11.49). Thus the Fourier Inversion Formula (11.76) implies that if $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$, then f can be modified on a set of measure zero to become a uniformly continuous function on $\mathbf{R}$.

The Fourier Inversion Formula now allows us to calculate the Fourier transform of P_y for each $y > 0$.

11.78 Example *Fourier transform of P_y*

Suppose $y > 0$. Define $f \colon \mathbf{R} \to (0, 1]$ by

$$f(t) = e^{-2\pi y|t|}.$$

Then $\hat{f} = P_y$ by 11.57. Hence both f and $\hat{f}$ are in $L^1(\mathbf{R})$. Thus we can apply the Fourier Inversion Formula (11.76), concluding that

11.79
$$(P_y)\hat{\ }(x) = (\hat{f})\hat{\ }(x) = f(-x) = e^{-2\pi y|x|}$$

for all $x \in \mathbf{R}$.

Now we can prove that the map on $L^1(\mathbf{R})$ defined by $f \mapsto \hat{f}$ is one-to-one.

11.80 ***functions are determined by their Fourier transforms***

Suppose $f \in \mathcal{L}^1(\mathbf{R})$ and $\hat{f}(t) = 0$ for every $t \in \mathbf{R}$. Then $f = 0$.

Proof Because $\hat{f} = 0$, we also have $(\hat{f})\hat{\ } = 0$. The Fourier Inversion Formula (11.76) now implies that $f = 0$. ■

The next result could be proved directly using the definition of convolution and Tonelli's/Fubini's Theorems. However, the following cute proof deserves to be seen.

11.81 ***convolution is associative***

Suppose $f, g, h \in L^1(\mathbf{R})$. Then $(f * g) * h = f * (g * h)$.

Proof The Fourier transform of $(f * g) * h$ and the Fourier transform of $f * (g * h)$ both equal $\hat{f}\hat{g}\hat{h}$ (by 11.66). Because the Fourier transform is a one-to-one mapping on $L^1(\mathbf{R})$ [see 11.80], this implies that $(f * g) * h = f * (g * h)$. ■

Extending Fourier Transform to $L^2(\mathbf{R})$

We now prove that the map $f \mapsto \hat{f}$ preserves $L^2(\mathbf{R})$ norms on $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$.

11.82 ***Plancherel's Theorem: Fourier transform preserves $L^2(\mathbf{R})$ norms***

Suppose $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then $\|\hat{f}\|_2 = \|f\|_2$.

Proof First consider the case where $\hat{f} \in L^1(\mathbf{R})$ in addition to the hypothesis that $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Define $g \colon \mathbf{R} \to \mathbf{C}$ by $g(x) = \overline{f(-x)}$. Then $\hat{g}(t) = \overline{\hat{f}(t)}$ for all $t \in \mathbf{R}$, as is easy to verify. Now

$$\|f\|_2^2 = \int_{-\infty}^{\infty} f(x)\overline{f(x)}\,dx$$

$$= \int_{-\infty}^{\infty} f(-x)\overline{f(-x)}\,dx$$

11.83
$$= \int_{-\infty}^{\infty} (\hat{f})\hat{\ }(x) g(x)\,dx$$

11.84
$$= \int_{-\infty}^{\infty} \hat{f}(x)\hat{g}(x)\,dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{f}(x)}\,dx$$

$$= \|\hat{f}\|_2^2,$$

where 11.83 holds by the Fourier Inversion Formula (11.76) and 11.84 follows from 11.59. The equation above shows that our desired result holds in the case when $\hat{f} \in L^1(\mathbf{R})$.

Now consider arbitrary $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. If $y > 0$, then $f * P_y \in L^1(\mathbf{R})$ by 11.64. If $x \in \mathbf{R}$, then

$$(f * P_y)\hat{}(x) = \hat{f}(x)(P_y)\hat{}(x)$$

11.85 $$= \hat{f}(x)e^{-2\pi y|x|},$$

where the first equality above comes from 11.66 and the second equality comes from 11.79. The equation above shows that $(f * P_y)\hat{} \in L^1(\mathbf{R})$. Thus we can apply the first case to $f * P_y$, concluding that

$$\|f * P_y\|_2 = \|(f * P_y)\hat{}\|_2.$$

As $y \downarrow 0$, the left side of the equation above converges to $\|f\|_2$ [by 11.74]. As $y \downarrow 0$, the right side of the equation above converges to $\|\hat{f}\|_2$ [by the explicit formula for $f * P_y$ given in 11.85 and the Monotone Convergence Theorem (3.11)]. Thus the equation above implies that $\|\hat{f}\|_2 = \|f\|_2$. ■

Because $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, Plancherel's Theorem (11.82) allows us to extend the map $f \mapsto \hat{f}$ uniquely to a bounded linear map from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ (see Exercise 14 in Section 6C). This extension is called the *Fourier transform* on $L^2(\mathbf{R})$; it gets its own notation, as shown below.

11.86 Definition ***Fourier transform on*** $L^2(\mathbf{R})$

The *Fourier transform* $\mathcal{F}$ on $L^2(\mathbf{R})$ is the bounded operator on $L^2(\mathbf{R})$ such that $\mathcal{F}f = \hat{f}$ for all $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$.

For $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we can use either $\hat{f}$ or $\mathcal{F}f$ to denote the Fourier transform of f. But if $f \in L^1(\mathbf{R}) \setminus L^2(\mathbf{R})$, we will use only the notation $\hat{f}$, and if $f \in L^2(\mathbf{R}) \setminus L^1(\mathbf{R})$, we will use only the notation $\mathcal{F}f$.

Suppose $f \in L^2(\mathbf{R}) \setminus L^1(\mathbf{R})$ and $t \in \mathbf{R}$. Do not make the mistake of thinking that $(\mathcal{F}f)(t)$ equals

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx.$$

Indeed, the integral above makes no sense because $\left|f(x)e^{-2\pi itx}\right| = |f(x)|$ and $f \notin L^1(\mathbf{R})$. Instead of defining $\mathcal{F}f$ via the equation above, $\mathcal{F}f$ must be defined as the limit in $L^2(\mathbf{R})$ of $(f_1)\hat{}, (f_2)\hat{}, \ldots$, where $f_1, f_2, \ldots$ is a sequence in $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ such that

$$\|f - f_n\|_2 \to 0 \text{ as } n \to \infty.$$

For example, one could take $f_n = f\chi_{[-n,n]}$ because $\|f - f\chi_{[-n,n]}\|_2 \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem (3.31).

Because $\mathcal{F}$ is obtained by continuously extending [in the norm of $L^2(\mathbf{R})$] the Fourier transform from $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ to $L^2(\mathbf{R})$, we know that $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L^2(\mathbf{R})$. In other words, $\mathcal{F}$ is an isometry on $L^2(\mathbf{R})$. The next result shows that even more is true.

11.87 ***properties of the Fourier transform on $L^2(\mathbf{R})$***

(a) $\mathcal{F}$ is a unitary operator on $L^2(\mathbf{R})$.

(b) $\mathcal{F}^4 = I$.

(c) $\operatorname{sp}(\mathcal{F}) = \{1, i, -1, -i\}$.

Proof First we prove (b). Suppose $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. If $y > 0$, then $P_y \in L^1(\mathbf{R})$ and hence 11.64 implies that

11.88
$$f * P_y \in L^1(\mathbf{R}) \cap L^2(\mathbf{R}).$$

Also,

11.89
$$(f * P_y)\hat{\ } \in L^1(\mathbf{R}) \cap L^2(\mathbf{R}),$$

as follows from the equation $(f * P_y)\hat{\ } = \hat{f} \cdot (P_y)\hat{\ }$ [see 11.66] and the observation that $\hat{f} \in L^\infty(\mathbf{R})$, $(P_y)\hat{\ } \in L^1(\mathbf{R})$ [see 11.49 and 11.79] and the observation that $\hat{f} \in L^2(\mathbf{R})$, $(P_y)\hat{\ } \in L^\infty(\mathbf{R})$ [see 11.82 and 11.49].

Now the Fourier Inversion Formula (11.76) as applied to $f * P_y$ (which is valid by 11.88 and 11.89) implies that

$$\mathcal{F}^4(f * P_y) = f * P_y.$$

Taking the limit in $L^2(\mathbf{R})$ of both sides of the equation above as $y \downarrow 0$, we have $\mathcal{F}^4 f = f$ (by 11.74), completing the proof of (b).

Plancherel's Theorem (11.82) tells us that $\mathcal{F}$ is an isometry on $\mathcal{L}^2(\mathbf{R})$. Part (a) implies that $\mathcal{F}$ is surjective. Because a surjective isometry is unitary (see 10.61), we conclude that $\mathcal{F}$ is unitary, completing the proof of (a).

The Spectral Mapping Theorem [see 10.40—take $p(z) = z^4$] and (b) imply that $\alpha^4 = 1$ for each $\alpha \in \operatorname{sp}(T)$. In other words, $\operatorname{sp}(T) \subset \{1, i, -1, -i\}$. However, 1, i, -1, $-i$ are all eigenvalues of $\mathcal{F}$ (see Example 11.51 and Exercises 2, 3, and 4) and thus are all in $\operatorname{sp}(T)$. Hence $\operatorname{sp}(T) = \{1, i, -1, -i\}$, completing the proof of (c). ■

EXERCISES 11C

1 Suppose $f \in L^1(\mathbf{R})$. Prove that $\|\hat{f}\|_\infty = \|f\|_1$ if and only if there exists $\zeta \in \partial\mathbf{D}$ and $t \in \mathbf{R}$ such that $\zeta f(x) e^{-itx} \geq 0$ for almost every $x \in \mathbf{R}$.

2 Suppose $f(x) = xe^{-\pi x^2}$ for all $x \in \mathbf{R}$. Show that $\hat{f} = -if$.

3 Suppose $f(x) = 4\pi x^2 e^{-\pi x^2} - e^{-\pi x^2}$ for all $x \in \mathbf{R}$. Show that $\hat{f} = -f$.

4 Find $f \in L^1(\mathbf{R})$ such that $f \neq 0$ and $\hat{f} = if$.

5 Prove that if p is a polynomial on $\mathbf{R}$ with complex coefficients and $f\colon \mathbf{R} \to \mathbf{C}$ is defined by $f(x) = p(x)e^{-\pi x^2}$, then there exists a polynomial q on $\mathbf{R}$ with complex coefficients such that $\deg q = \deg p$ and $\hat{f}(t) = q(t)e^{-\pi t^2}$ for all $t \in \mathbf{R}$.

6 Suppose

$$f(x) = \begin{cases} xe^{-2\pi x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that $\hat{f}(t) = \dfrac{1}{4\pi^2(1+it)^2}$ for all $t \in \mathbf{R}$.

7 Prove the formulas in 11.55 for the Fourier transforms of translations, rotations, and dilations.

8 Suppose $f \in L^1(\mathbf{R})$ and $n \in \mathbf{Z}^+$. Define $g\colon \mathbf{R} \to \mathbf{C}$ by $g(x) = x^n f(x)$. Prove that if $g \in L^1(\mathbf{R})$, then $\hat{f}$ is n times continuously differentiable on $\mathbf{R}$ and

$$(\hat{f})^{(n)}(t) = (-2\pi i)^n \hat{g}(t)$$

for all $t \in \mathbf{R}$.

9 Suppose $n \in \mathbf{Z}^+$ and $f \in L^1(\mathbf{R})$ is n times continuously differentiable and $f^{(n)} \in L^1(\mathbf{R})$. Prove that if $t \in \mathbf{R}$, then

$$(f^{(n)})\hat{}\,(t) = (2\pi it)^n \hat{f}(t).$$

10 Suppose $1 \leq p \leq \infty$, $f \in L^p(\mathbf{R})$, and $g \in L^{p'}(\mathbf{R})$. Prove that $f * g$ is a uniformly continuous function on $\mathbf{R}$.

11 Suppose $f \in \mathcal{L}^\infty(\mathbf{R})$, $x \in \mathbf{R}$, and f is continuous at x. Prove that

$$\lim_{y \downarrow 0} (\mathcal{P}_y f)(x) = f(x).$$

12 Suppose $p \in [1, \infty]$ and $f \in L^p(\mathbf{R})$. Prove that $\mathcal{P}_y(\mathcal{P}_{y'} f) = \mathcal{P}_{y+y'} f$ for all $y, y' > 0$.

13 Suppose $p \in [1, \infty]$ and $f \in L^p(\mathbf{R})$. Prove that if $0 < y < y'$, then

$$\|\mathcal{P}_y f\|_p \geq \|\mathcal{P}_{y'} f\|_p.$$

14 Suppose $f \in L^1(\mathbf{R})$.

(a) Prove that $(\overline{f})\hat{}\,(t) = \overline{\hat{f}(-t)}$ for all $t \in \mathbf{R}$.

(b) Prove that $f(x) \in \mathbf{R}$ for almost every $x \in \mathbf{R}$ if and only if $\hat{f}(t) = \overline{\hat{f}(-t)}$ for all $t \in \mathbf{R}$.

15 Define $f \in L^1(\mathbf{R})$ by $f(x) = e^{-x^4}\chi_{[0,\infty)}(x)$. Show that $\hat{f} \notin L^1(\mathbf{R})$.

16 Suppose $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$. Prove that $f \in L^2(\mathbf{R})$ and $\hat{f} \in L^2(\mathbf{R})$.

17 Prove there exists a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $\displaystyle\lim_{t\to\pm\infty} g(t) = 0$ and $g \notin \{\hat{f} : f \in L^1(\mathbf{R})\}$.

18 Prove that if $f \in L^1(\mathbf{R})$, then $\|\hat{f}\|_2 = \|f\|_2$.
[*This exercise slightly improves Plancherel's Theorem (11.82) because here we have the weaker hypothesis that* $f \in L^1(\mathbf{R})$ *instead of* $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. *Because of Plancherel's Theorem, here you need only prove that if* $f \in L^1(\mathbf{R})$ *and* $\|f\|_2 = \infty$, *then* $\|\hat{f}\|_2 = \infty$.]

19 Suppose $y > 0$. Define on operator T on $L^2(\mathbf{R})$ by $Tf = f * P_y$.

(a) Show that T is a self-adjoint operator on $L^2(\mathbf{R})$.

(b) Show that $\operatorname{sp}(T) = [0, 1]$.

[*Because the spectrum of each compact operator is a countable set (by 10.93), part (b) above implies that T is not a compact operator. This conclusion differs from the situation on the unit circle—see Exercise 9 in Section 11B.*]

20 Prove that if $f \in L^1(\mathbf{R})$ and $g \in L^2(\mathbf{R})$, then $\mathcal{F}(f * g) = \hat{f} \cdot \mathcal{F}g$.

21 Prove that $f, g \in L^2(\mathbf{R})$, then $(fg)\hat{} = (\mathcal{F}f) * (\mathcal{F}g)$.

Chapter 12
Probability Measures

Probability theory has become increasingly important in multiple parts of science. Getting deeply into probability theory requires a full book, not just a chapter. For readers who intend to pursue further studies in probability theory, this chapter gives you a good head start. For readers not intending to delve further into probability theory, this chapter gives you a taste of the subject.

Modern probability theory makes major use of measure theory. As we will see, a probability measure is simply a measure such that the measure of the whole space equals 1. Thus a thorough understanding of the chapters of this book dealing with measure theory and integration provides a solid foundation for probability theory.

However, probability theory is not simply the special case of measure theory where the whole space has measure 1. The questions that probability theory investigates differ from the questions natural to measure theory. For example, the probability notions of independent sets and independent random variables, which are introduced in this chapter, do not arise in measure theory.

Even when concepts in probability theory have the same meaning as well-known concepts in measure theory, the terminology and notation can be quite different. Thus one goal of this chapter is to introduce the vocabulary of probability theory. This difference in vocabulary between probability theory and measure theory occurred because the two subjects had different historical developments, only coming together in the first half of the twentieth century.



Dice used in games of chance. The beginning of probability theory can be traced to correspondence in 1654 between Pierre de Fermat (1601–1665) and Blaise Pascal (1623–1662) about how to distribute fairly money bet on an unfinished game of dice.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_12

Probability Spaces

We begin with an intuitive and nonrigorous motivation. Suppose we pick a real number at random from the interval $(0,1)$, with each real number having an equal probability of being chosen (whatever that means). What is the probability that the chosen number is in the interval $(\frac{9}{10},1)$? The only reasonable answer to this question is $\frac{1}{10}$. More generally, if $I_1, I_2, \ldots$ is a disjoint sequence of open intervals contained in $(0,1)$, then the probability that our randomly chosen real number is in $\bigcup_{n=1}^{\infty} I_n$ should be $\sum_{n=1}^{\infty} \ell(I_n)$, where $\ell(I)$ denotes the length of an interval I. Still more generally, if A is a Borel subset of $(0,1)$, then the probability that our random number is in A should be the Lebesgue measure of A.

With the paragraph above as motivation, we are now ready to define a probability measure. We will use the notation and terminology common in probability theory instead of the conventions of measure theory.

In particular, the set in which everything takes place is now called Ω instead of the usual X in measure theory. The σ-algebra on Ω is called $\mathcal{F}$ instead of $\mathcal{S}$, which we have used in previous chapters. Our measure is now called P instead of μ. This new notation and terminology can be disorienting when first encountered. However, reading this chapter should help you become comfortable with this notation and terminology, which are standard in probability theory.

12.1 Definition ***probability measure***

Suppose $\mathcal{F}$ is a σ-algebra on a set Ω.

- A *probability measure* on $(\Omega, \mathcal{F})$ is a measure P on $(\Omega, \mathcal{F})$ such that $P(\Omega) = 1$.
- Ω is called the *sample space*.
- An *event* is an element of $\mathcal{F}$ ($\mathcal{F}$ need not be mentioned if it is clear from the context).
- If A is an event, then $P(A)$ is called the *probability* of A.
- If P is a probability measure on $(\Omega, \mathcal{F})$, then the triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

12.2 Example *probability measures*

- Suppose $n \in \mathbf{Z}^+$ and Ω is a set containing exactly n elements. Let $\mathcal{F}$ denote the collection of all subsets of Ω. Then

$$\frac{\text{counting measure on } \Omega}{n}$$

is a probability measure on $(\Omega, \mathcal{F})$.

- As a more specific example of the previous item, suppose that $\Omega = \{40, 41, \ldots, 49\}$ and $P =$ (counting measure on Ω)$/10$. Let $A = \{\omega \in \Omega : \omega \text{ is even}\}$ and $B = \{\omega \in \Omega : \omega \text{ is prime}\}$. Then $P(A)$ [which is the probability that an element of this sample space Ω is even] is $\frac{1}{2}$ and $P(B)$ [which is the probability that an element of this sample space Ω is prime] is $\frac{3}{10}$.

This example illustrates the common practice in probability theory of using lower case ω to denote a typical element of upper case Ω.

- Let λ denote Lebesgue measure on the interval $[0,1]$. Then λ is a probability measure on $([0,1], \mathcal{B})$, where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of $[0,1]$.

- Let λ denote Lebesgue measure on $\mathbf{R}$, and let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbf{R}$. Define $h \colon \mathbf{R} \to (0, \infty)$ by $h(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then $h\,d\lambda$ is a probability measure on $(\mathbf{R}, \mathcal{B})$ [see 9.6 for the definition of $h\,d\lambda$].

In measure theory, we used the notation χ_A to denote the characteristic function of a set A. In probability theory, this function has a different name and different notation, as we see in the next definition.

12.3 Definition *indicator function;* 1_A

If Ω is a set and $A \subset \Omega$, then the *indicator function* of A is the function $1_A \colon \Omega \to \mathbf{R}$ defined by

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

The next definition gives the replacement in probability theory for measure theory's phrase *almost every*.

12.4 Definition *almost surely*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. An event A is said to happen *almost surely* if the probability of A is 1, or equivalently if $P(\Omega \setminus A) = 0$.

12.5 Example *almost surely*

Let P denote Lebesgue measure on the interval $[0,1]$. If $\omega \in [0,1]$, then ω is almost surely an irrational number (because the set of rational numbers has Lebesgue measure 0).

This example shows that an event having probability 1 (equivalent to happening almost surely) does not mean that the event definitely happens. Conversely, an event having probability 0 does not mean that the event is impossible. Specifically, if a real number is chosen at random from $[0,1]$ using Lebesgue measure as the probability, then the probability that the number is rational is 0, but that event can still happen.

The following result is frequently useful in probability theory. A careful reading of the proof of this result, as our first proof in this chapter, should give you good practice using some of the notation and terminology commonly used in probability theory. This proof also illustrates the point that having a good understanding of measure theory and integration can often be extremely useful in probability theory—here we use the Monotone Convergence Theorem.

12.6 *Borel–Cantelli Lemma*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A_1, A_2, \dots$ is a sequence of events such that $\sum_{n=1}^{\infty} P(A_n) < \infty$. Then

$$P(\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n \in \mathbf{Z}^+\}) = 0.$$

Proof Let $A = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n \in \mathbf{Z}^+\}$. Then

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

Thus $A \in \mathcal{F}$, and hence $P(A)$ makes sense.

The Monotone Convergence Theorem (3.11) implies that

$$\int_{\Omega} \Bigl(\sum_{n=1}^{\infty} 1_{A_n} \Bigr) \, dP = \sum_{n=1}^{\infty} \int_{\Omega} 1_{A_n} \, dP = \sum_{n=1}^{\infty} P(A_n) < \infty.$$

Thus $\sum_{n=1}^{\infty} 1_{A_n}$ is almost surely finite. Hence $P(A) = 0$. ■

Independent Events and Independent Random Variables

The notion of independent events, which we now define, is one of the key concepts that distinguishes probability theory from measure theory.

12.7 Definition *independent events*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

- Two events A and B are called *independent* if

$$P(A \cap B) = P(A) \cdot P(B).$$

- More generally, a family of events $\{A_k\}_{k \in \Gamma}$ is called *independent* if

$$P(A_{k_1} \cap \cdots \cap A_{k_n}) = P(A_{k_1}) \cdots P(A_{k_n})$$

  whenever $k_1, \dots, k_n$ are distinct elements of Γ.

The next two examples should help develop your intuition about independent events.

12.8 **Example** *independent events: coin tossing*

Suppose $\Omega = \{H, T\}^4$, where H and T are symbols that you can think of as denoting "heads" and "tails". Thus elements of Ω are 4-tuples of the form

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4),$$

where each ω_j is H or T. Let $\mathcal{F}$ be the collection of all subsets of Ω, and let $P = (\text{counting measure on } \Omega)/16$, as we expect from a fair coin toss.

Let

$$A = \{\omega \in \Omega : \omega_1 = \omega_2 = \omega_3 = H\} \quad \text{and} \quad B = \{\omega \in \Omega : \omega_4 = H\}.$$

Then A contains two elements and thus $P(A) = \frac{1}{8}$, corresponding to probability $\frac{1}{8}$ that the first three coin tosses are all heads. Also, B contains eight elements and thus $P(B) = \frac{1}{2}$, corresponding to probability $\frac{1}{2}$ that the fourth coin toss is heads.

Now

$$P(A \cap B) = \tfrac{1}{16} = P(A) \cdot P(B),$$

where the first equality holds because $A \cap B$ consists of only the one element (H, H, H, H) and the second equality holds because $P(A) = \frac{1}{8}$ and $P(B) = \frac{1}{2}$. The equation above shows that A and B are independent events.

If we toss a fair coin many times, we expect that about half the time it will be heads. Thus some people mistakenly believe that if the first three tosses of a fair coin are heads, then the fourth toss should have a higher probability of being tails, to balance out the previous heads. However, the coin cannot remember that it had three heads in a row, and thus the fourth coin toss has probability $\frac{1}{2}$ of being heads regardless of the results of the three previous coin tosses. The independence of the events A and B above captures the notion that the results of a fair coin toss do not depend upon previous results.

12.9 **Example** *independent events: product probability space*

Suppose $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ are probability spaces. Then

$$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \times P_2),$$

as defined in Chapter 5, is also a probability space.

If $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, then $(A \times \Omega_2) \cap (\Omega_1 \times B) = A \times B$. Thus

$$\begin{aligned}(P_1 \times P_2)\big((A \times \Omega_2) \cap (\Omega_1 \times B)\big) &= (P_1 \times P_2)(A \times B)\\ &= P_1(A) \cdot P_2(B)\\ &= (P_1 \times P_2)(A \times \Omega_2) \cdot (P_1 \times P_2)(\Omega_1 \times B),\end{aligned}$$

where the second equality follows from the definition of the product measure, and the third equality holds because of the definition of the product measure and because P_1 and P_2 are probability measures.

The equation above shows that the events $A \times \Omega_2$ and $\Omega_1 \times B$ are independent events in $\mathcal{F}_1 \otimes \mathcal{F}_2$.

Compare the next result to the Borel–Cantelli Lemma (12.6).

12.10 *relative of Borel–Cantelli Lemma*

Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space and $\{A_n\}_{n\in\mathbf{Z}^+}$ is an independent family of events such that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Then

$$P(\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n \in \mathbf{Z}^+\}) = 1.$$

Proof Let $A = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n \in \mathbf{Z}^+\}$. Then

12.11
$$\Omega \setminus A = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} (\Omega \setminus A_n).$$

If $m, M \in \mathbf{Z}^+$ are such that $m \leq M$, then

$$\begin{aligned} P\Big(\bigcap_{n=m}^{M} (\Omega \setminus A_n)\Big) &= \prod_{n=m}^{M} P(\Omega \setminus A_n) \\ &= \prod_{n=m}^{M} \big(1 - P(A_n)\big) \\ &\leq e^{-\sum_{n=m}^{M} P(A_n)}, \end{aligned}$$

12.12

where the first line holds because the family $\{\Omega \setminus A_n\}_{n\in\mathbf{Z}^+}$ is independent (see Exercise 4) and the third line holds because $1 - t \leq e^{-t}$ for all $t \geq 0$.

Because $\sum_{n=1}^{\infty} P(A_n) = \infty$, by choosing M large we can make the right side of 12.12 as close to 0 as we wish. Thus

$$P\Big(\bigcap_{n=m}^{\infty} (\Omega \setminus A_n)\Big) = 0$$

for all $m \in \mathbf{Z}^+$. Now 12.11 implies that $P(\Omega \setminus A) = 0$. Thus we conclude that $P(A) = 1$, as desired. ∎

For the rest of this chapter, assume that $\mathbf{F} = \mathbf{R}$. Thus, for example, if $(\Omega, \mathcal{F}, P)$ is a probability space, then $\mathcal{L}^1(P)$ will always refer to the vector space of ***real-valued*** $\mathcal{F}$-measurable functions on Ω such that $\int_\Omega |f| \, dP < \infty$.

12.13 Definition *random variable; expectation;* EX

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

- A *random variable* on $(\Omega, \mathcal{F})$ is a measurable function from Ω to $\mathbf{R}$.
- If $X \in \mathcal{L}^1(P)$, then the *expectation* (sometimes called the *expected value*) of the random variable X is denoted EX and is defined by

$$EX = \int_\Omega X \, dP.$$

If $\mathcal{F}$ is clear from the context, the phrase "random variable on Ω" can be used instead of the more precise phrase "random variable on $(\Omega, \mathcal{F})$". If both Ω and $\mathcal{F}$ are clear from the context, then the phrase "random variable" has no ambiguity and is often used.

Because $P(\Omega) = 1$, the expectation EX of a random variable $X \in \mathcal{L}^1(P)$ can be thought of as the *average* or *mean value* of X.

The next definition illustrates a convention often used in probability theory: the variable is often omitted when describing a set. Thus, for example, $\{X \in U\}$ means $\{\omega \in \Omega : X(\omega) \in U\}$, where U is a subset of $\mathbf{R}$. Also, probabilists often also omit the set brackets, as we do for the first time in the second bullet point below, when appropriate parentheses are nearby.

12.14 Definition ***independent random variables***

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

- Two random variables X and Y are called *independent* if $\{X \in U\}$ and $\{Y \in V\}$ are independent events for all Borel sets U, V in $\mathbf{R}$.
- More generally, a family of random variables $\{X_k\}_{k \in \Gamma}$ is called *independent* if $\{X_k \in U_k\}_{k \in \Gamma}$ is independent for all families of Borel sets $\{U_k\}_{k \in \Gamma}$ in $\mathbf{R}$.

12.15 Example *independent random variables*

- Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$. Then 1_A and 1_B are independent random variables if and only if A and B are independent events, as you should verify.
- Suppose $\Omega = \{H, T\}^4$ is the sample space of four coin tosses, with Ω and P as in Example 12.8. Define random variables X and Y by

$$X(\omega_1, \omega_2, \omega_3, \omega_4) = \text{number of } \omega_1, \omega_2, \omega_3 \text{ that equal } H$$

and

$$Y(\omega_1, \omega_2, \omega_3, \omega_4) = \text{number of } \omega_3, \omega_4 \text{ that equal } H.$$

Then X and Y are not independent random variables because $P(X = 3) = \frac{1}{8}$ and $P(Y = 0) = \frac{1}{4}$ but $P(\{X = 3\} \cap \{Y = 0\}) = P(\varnothing) = 0 \neq \frac{1}{8} \cdot \frac{1}{4}$.

- Suppose $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ are probability spaces, Z_1 is a random variable on Ω_1, and Z_2 is a random variable on Ω_2. Define random variables X and Y on $\Omega_1 \times \Omega_2$ by

$$X(\omega_1, \omega_2) = Z_1(\omega_1) \quad \text{and} \quad Y(\omega_1, \omega_2) = Z_2(\omega_2).$$

Then X and Y are independent random variables on $\Omega_1 \times \Omega_2$ (with respect to the probability measure $P_1 \times P_2$), as you should verify.

If X is a random variable and $f : \mathbf{R} \to \mathbf{R}$ is Borel measurable, then $f \circ X$ is a random variable (by 2.44). For example, if X is a random variable, then X^2 and e^X are random variables. The next result states that compositions preserve independence.

12.16 ***functions of independent random variables are independent***

Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, X and Y are independent random variables, and $f, g\colon \mathbf{R} \to \mathbf{R}$ are Borel measurable. Then $f \circ X$ and $g \circ Y$ are independent random variables.

Proof Suppose U, V are Borel subsets of $\mathbf{R}$. Then

$$\begin{aligned} P\bigl(\{f \circ X \in U\} \cap \{g \circ Y \in V\}\bigr) &= P\bigl(\{X \in f^{-1}(U)\} \cap \{Y \in g^{-1}(V)\}\bigr) \\ &= P\bigl(X \in f^{-1}(U)\bigr) \cdot P\bigl(Y \in g^{-1}(V)\bigr) \\ &= P(f \circ X \in U) \cdot P(g \circ Y \in V), \end{aligned}$$

where the second equality holds because X and Y are independent random variables. The equation above shows that $f \circ X$ and $g \circ Y$ are independent random variables. ∎

If $X, Y \in \mathcal{L}^1(P)$, then clearly $E(X+Y) = E(X) + E(Y)$. The next result gives a nice formula for the expectation of XY when X and Y are independent. This formula has sometimes been called the dream equation of calculus students.

12.17 ***expectation of product of independent random variables***

Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space and X and Y are independent random variables in $\mathcal{L}^2(P)$. Then

$$E(XY) = EX \cdot EY.$$

Proof First consider the case where X and Y are each simple functions, taking on only finitely many values. Thus there are distinct numbers $a_1, \ldots, a_M \in \mathbf{R}$ and distinct numbers $b_1, \ldots, b_N \in \mathbf{R}$ such that

$$X = a_1 1_{\{X = a_1\}} + \cdots + a_M 1_{\{X = a_M\}} \quad \text{and} \quad Y = b_1 1_{\{Y = b_1\}} + \cdots + b_N 1_{\{Y = b_N\}}.$$

Now

$$XY = \sum_{j=1}^{M} \sum_{k=1}^{N} a_j b_k 1_{\{X = a_j\}} 1_{\{Y = b_k\}} = \sum_{j=1}^{M} \sum_{k=1}^{N} a_j b_k 1_{\{X = a_j\} \cap \{Y = b_k\}}.$$

Thus

$$\begin{aligned} E(XY) &= \sum_{j=1}^{M} \sum_{k=1}^{N} a_j b_k P\bigl(\{X = a_j\} \cap \{Y = b_k\}\bigr) \\ &= \Bigl(\sum_{j=1}^{M} a_j P(X = a_j)\Bigr) \Bigl(\sum_{k=1}^{N} b_k P(Y = b_k)\Bigr) \\ &= EX \cdot EY, \end{aligned}$$

where the second equality above comes from the independence of X and Y. The last equation gives the desired conclusion in the case where X and Y are simple functions.

Now consider arbitrary independent random variables X and Y in $\mathcal{L}^2(P)$. Let $f_1, f_2, \ldots$ be a sequence of Borel measurable simple functions from $\mathbf{R}$ to $\mathbf{R}$ that approximate the identity function on $\mathbf{R}$ (the function $t \mapsto t$) in the sense that $\lim_{n\to\infty} f_n(t) = t$ for every $t \in \mathbf{R}$ and $|f_n(t)| \le t$ for all $t \in \mathbf{R}$ and all $n \in \mathbf{Z}^+$ (see 2.89, taking f to be the identity function, for construction of this sequence). The random variables $f_n \circ X$ and $f_n \circ Y$ are independent (by 12.17). Thus the result in the first paragraph of this proof shows that

$$E\big((f_n \circ X)(f_n \circ Y)\big) = E(f_n \circ X) \cdot E(f_n \circ Y)$$

for each $n \in \mathbf{Z}^+$. The limit as $n \to \infty$ of the right side of the equation above equals $EX \cdot EY$ [by the Dominated Convergence Theorem (3.31)]. The limit as $n \to \infty$ of the left side of the equation above equals $E(XY)$ [use Hölder's inequality (7.9)]. Thus the equation above implies that $E(XY) = EX \cdot EY$. ■

Variance and Standard Deviation

The variance and standard deviation of a random variable, defined below, measure how much a random variable differs from its expectation.

12.18 Definition *variance; standard deviation;* $\sigma(X)$

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $X \in \mathcal{L}^2(P)$ is a random variable.

- The *variance* of X is defined to be $E\big((X - EX)^2\big)$.
- The *standard deviation* of X is denoted $\sigma(X)$ and is defined by

  $$\sigma(X) = \sqrt{E\big((X - EX)^2\big)}.$$

  In other words, the standard deviation of X is the square root of the variance of X.

The notation $\sigma^2(X)$ means $\big(\sigma(X)\big)^2$. Thus $\sigma^2(X)$ is the variance of X.

12.19 Example *variance and standard deviation of an indicator function*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A \in \mathcal{F}$ is an event. Then

$$\begin{aligned}
\sigma^2(1_A) &= E\big((1_A - E1_A)^2\big) \\
&= E\big((1_A - P(A))^2\big) \\
&= E\big(1_A - 2P(A) \cdot 1_A + P(A)^2\big) \\
&= P(A) - 2\big(P(A)\big)^2 + \big(P(A)\big)^2 \\
&= P(A) \cdot \big(1 - P(A)\big).
\end{aligned}$$

Thus $\sigma(1_A) = \sqrt{P(A) \cdot \big(1 - P(A)\big)}$.

The next result gives a formula for the variance of a random variable. This formula is often more convenient to use than the formula that defines the variance.

12.20 *variance formula*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $X \in \mathcal{L}^2(P)$ is a random variable. Then

$$\sigma^2(X) = E(X^2) - (EX)^2.$$

Proof We have

$$\begin{aligned}\sigma^2(X) &= E\big((X - EX)^2\big)\\ &= E\big(X^2 - 2(EX)X + (EX)^2\big)\\ &= E(X^2) - 2(EX)^2 + (EX)^2\\ &= E(X^2) - (EX)^2,\end{aligned}$$

as desired. ■

Our next result is called Chebyshev's inequality. It states, for example (take $t = 2$ below) that the probability that a random variable X differs from its average by more than twice its standard deviation is at most $\frac{1}{4}$. Note that $P\big(|X - EX| \geq t\sigma(X)\big)$ is shorthand for $P\big(\{\omega \in \Omega : |X(\omega) - EX| \geq t\sigma(X)\}\big)$.

12.21 *Chebyshev's inequality*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $X \in \mathcal{L}^2(P)$ is a random variable. Then

$$P\big(|X - EX| \geq t\sigma(X)\big) \leq \frac{1}{t^2}$$

for all $t > 0$.

Proof Suppose $t > 0$. Then

$$\begin{aligned}P\big(|X - EX| \geq t\sigma(X)\big) &= P\big(|X - EX|^2 \geq t^2\sigma^2(X)\big)\\ &\leq \frac{1}{t^2\sigma^2(X)} E\big((X - EX)^2\big)\\ &= \frac{1}{t^2},\end{aligned}$$

where the second line above comes from applying Markov's inequality (4.1) with $h = |X - EX|^2$ and $c = t^2\sigma^2(X)$. ■

The next result gives a beautiful formula for the variance of the sum of independent random variables.

12.22 ***variance of sum of independent random variables***

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $X_1, \dots, X_n \in \mathcal{L}^2(P)$ are independent random variables. Then

$$\sigma^2(X_1 + \cdots + X_n) = \sigma^2(X_1) + \cdots + \sigma^2(X_n).$$

Proof Using the variance formula given by 12.20, we have

$$\begin{aligned}
\sigma^2\Big(\sum_{k=1}^n X_k\Big) &= E\Big(\big(\sum_{k=1}^n X_k\big)^2\Big) - \Big(E\big(\sum_{k=1}^n X_k\big)\Big)^2 \\
&= E\Big(\sum_{k=1}^n X_k{}^2\Big) + 2E\Big(\sum_{1\le j<k\le n} X_j X_k\Big) - \Big(\sum_{k=1}^n EX_k\Big)^2 \\
&= \sum_{k=1}^n E(X_k^2) - \sum_{k=1}^n (EX_k)^2 + 2\Big(\sum_{1\le j<k\le n} E(X_j X_k)\Big) - 2\Big(\sum_{1\le j<k\le n} EX_j \cdot EX_k\Big) \\
&= \sum_{k=1}^n \sigma^2(X_k),
\end{aligned}$$

where the last equality uses 12.20, 12.17, and the hypothesis that $X_1, \dots, X_n$ are independent random variables. ■

Conditional Probability and Bayes' Theorem

The conditional probability $P_B(A)$ that we are about to define should be interpreted to mean the probability that ω will be in A given that $\omega \in B$. Because ω is in $A \cap B$ if and only if $\omega \in B$ and $\omega \in A$, and because we expect probabilities to multiply, it is reasonable to expect that

$$P(B) \cdot P_B(A) = P(A \cap B).$$

Thus we are led to the following definition.

12.23 Definition ***conditional probability; P_B***

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and B is an event with $P(B) > 0$. Define $P_B \colon \mathcal{F} \to [0, 1]$ by

$$P_B(A) = \frac{P(A \cap B)}{P(B)}.$$

If $A \in \mathcal{F}$, then $P_B(A)$ is called the *conditional probability* of A given B.

You should verify that with B as above, P_B is a probability measure on $(\Omega, \mathcal{F})$. If $A \in \mathcal{F}$, then $P_B(A) = P(A)$ if and only if A and B are independent events.

We now present two versions of what is called Bayes' Theorem. You should do a web search and read about the many uses of these results, including some controversial applications.

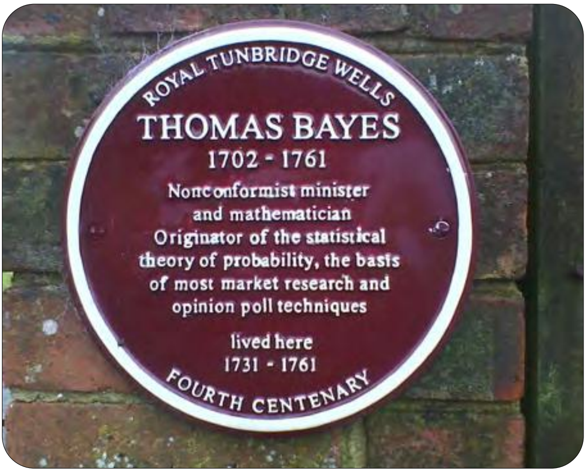
12.24 *Bayes' Theorem, first version*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and A, B are events with positive probability. Then

$$P_B(A) = \frac{P_A(B) \cdot P(A)}{P(B)}.$$

Proof We have

$$P_B(A) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B) \cdot P(A)}{P(A) \cdot P(B)} = \frac{P_A(B) \cdot P(A)}{P(B)}.$$



Plaque honoring Thomas Bayes in Tunbridge Wells, England.

12.25 *Bayes' Theorem, second version*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, B is an event with positive probability, and $A_1, \dots, A_n$ are pairwise disjoint events, each with positive probability, such that $A_1 \cup \dots \cup A_n = \Omega$. Then

$$P_B(A_k) = \frac{P_{A_k}(B) \cdot P(A_k)}{\sum_{j=1}^n P_{A_j}(B) \cdot P(A_j)}$$

for each $k \in \{1, \dots, n\}$.

Proof Consider the denominator of the expression above. We have

12.26 $$\sum_{j=1}^n P_{A_j}(B) \cdot P(A_j) = \sum_{j=1}^n P(A_j \cap B) = P(B).$$

Now suppose $k \in \{1, \dots, n\}$. Then

$$P_B(A_k) = \frac{P_{A_k}(B) \cdot P(A_k)}{P(B)} = \frac{P_{A_k}(B) \cdot P(A_k)}{\sum_{j=1}^n P_{A_j}(B) \cdot P(A_j)},$$

where the first equality comes from the first version of Bayes's Theorem (12.24) and the second equality comes from 12.26.

Distribution and Density Functions of Random Variables

For the rest of this chapter, let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbf{R}$.

Each random variable X determines a probability measure P_X on $(\mathbf{R}, \mathcal{B})$ and a function $\tilde{X}\colon \mathbf{R} \to [0,1]$ as in the next definition.

12.27 Definition ***probability distribution and distribution function; P_X; $\tilde{X}$***

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and X is a random variable.

- The *probability distribution* of X is the probability measure P_X defined on $(\mathbf{R}, \mathcal{B})$ by
$$P_X(B) = P(X \in B) = P\big(X^{-1}(B)\big).$$
- The *distribution function* of X is the function $\tilde{X}\colon \mathbf{R} \to [0,1]$ defined by
$$\tilde{X}(s) = P_X\big((-\infty, s]\big) = P(X \le s).$$

You should verify that the probability distribution P_X as defined above is indeed a probability measure on $(\mathbf{R}, \mathcal{B})$. Note that the distribution function $\tilde{X}$ depends upon the probability measure P as well as the random variable X, even though P is not included in the notation $\tilde{X}$ (because P is usually clear from the context).

12.28 Example *probability distribution and distribution function of an indicator function*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A \in \mathcal{F}$ is an event. Then you should verify that
$$P_{1_A} = \big(1 - P(A)\big)\delta_0 + P(A)\delta_1,$$
where for $t \in \mathbf{R}$ the measure δ_t on $(\mathbf{R}, \mathcal{B})$ is defined by
$$\delta_t(B) = \begin{cases} 1 & \text{if } t \in B, \\ 0 & \text{if } t \notin B. \end{cases}$$

The distribution function of 1_A is the function $(1_A)\tilde{}\colon \mathbf{R} \to [0,1]$ given by
$$(1_A)\tilde{}(s) = \begin{cases} 0 & \text{if } s < 0, \\ 1 - P(A) & \text{if } 0 \le s < 1, \\ 1 & \text{if } s \ge 1, \end{cases}$$
as you should verify.

One direction of the next result states that every probability distribution is a right-continuous increasing function, with limit 0 at $-\infty$ and limit 1 at ∞. The other direction of the next result states that every function with those properties is the distribution function of some random variable on some probability space. The proof shows that we can take the sample space to be $(0,1)$, the σ-algebra to be the Borel subsets of $(0,1)$, and the probability measure to be Lebesgue measure on $(0,1)$.

Your understanding of the proof of the next result should be enhanced by Exercise 13, which asserts that if the function $H\colon \mathbf{R} \to (0,1)$ appearing in the next result is continuous and injective, then the random variable $X\colon (0,1) \to \mathbf{R}$ in the proof is the inverse function of H.

12.29 *characterization of distribution functions*

Suppose $H\colon \mathbf{R} \to [0,1]$ is a function. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a random variable X on $(\Omega, \mathcal{F})$ such that $H = \tilde{X}$ if and only if the following conditions are all satisfied:

(a) $s < t \Rightarrow H(s) \leq H(t)$ (in other words, H is an increasing function);

(b) $\lim\limits_{t \to -\infty} H(t) = 0$;

(c) $\lim\limits_{t \to \infty} H(t) = 1$;

(d) $\lim\limits_{t \downarrow s} H(t) = H(s)$ for every $s \in \mathbf{R}$ (in other words, H is right continuous).

Proof First suppose $H = \tilde{X}$ for some probability space $(\Omega, \mathcal{F}, P)$ and some random variable X on $(\Omega, \mathcal{F})$. Then (a) holds because $s < t$ implies $(-\infty, s] \subset (-\infty, t]$. Also, (b) and (d) follow from 2.60. Furthermore, (c) follows from 2.59, completing the proof in this direction.

To prove the other direction, now suppose that H satisfies (a) through (d). Let $\Omega = (0,1)$, let $\mathcal{F}$ be the collection of Borel subsets of the interval $(0,1)$, and let P be Lebesgue measure on $\mathcal{F}$. Define a random variable X by

12.30 $$X(\omega) = \sup\{t \in \mathbf{R} : H(t) < \omega\}$$

for $\omega \in (0,1)$. Clearly X is an increasing function and thus is measurable (in other words, X is indeed a random variable).

Suppose $s \in \mathbf{R}$. If $\omega \in \big(0, H(s)\big]$, then

$$X(\omega) \leq X\big(H(s)\big) = \sup\{t \in \mathbf{R} : H(t) < H(s)\} \leq s,$$

where the first inequality holds because X is an increasing function and the last inequality holds because H is an increasing function. Hence

12.31 $$\big(0, H(s)\big] \subset \{X \leq s\}.$$

If $\omega \in (0,1)$ and $X(\omega) \leq s$, then $H(t) \geq \omega$ for all $t > s$ (by 12.30). Thus

$$H(s) = \lim_{t \downarrow s} H(t) \geq \omega,$$

where the equality above comes from (d). Rewriting the inequality above, we have $\omega \in \big(0, H(s)\big]$. Thus we have shown that $\{X \leq s\} \subset \big(0, H(s)\big]$, which when combined with 12.31 shows that $\{X \leq s\} = \big(0, H(s)\big]$. Hence

$$\tilde{X}(s) = P(X \leq s) = P\Big(\big(0, H(s)\big]\Big) = H(s),$$

as desired. ■

In the definition below and in the following discussion, λ denotes Lebesgue measure on $\mathbf{R}$, as usual.

12.32 Definition ***density function***

Suppose X is a random variable on some probability space. If there exists $h \in L^1(\mathbf{R})$ such that

$$\tilde{X}(s) = \int_{-\infty}^{s} h \, d\lambda$$

for all $s \in \mathbf{R}$, then h is called the *density function* of X.

If there is a density function of a random variable X, then it is unique [up to changes on sets of Lebesgue measure 0, which is already taken into account because we are thinking of density functions as elements of $L^1(\mathbf{R})$ instead of elements of $\mathcal{L}^1(\mathbf{R})$]; see Exercise 6 in Chapter 4.

If X is a random variable that has a density function h, then the distribution function $\tilde{X}$ is differentiable almost everywhere (with respect to Lebesgue measure) and $\tilde{X}'(s) = h(s)$ for almost every $s \in \mathbf{R}$ (by the second version of the Lebesgue Differentiation Theorem; see 4.19). Because $\tilde{X}$ is an increasing function, this implies that $h(s) \geq 0$ for almost every $s \in \mathbf{R}$. In other words, we can assume that a density function is nonnegative.

In the definition above of a density function, we started with a probability space and a random variable on it. Often in probability theory, the procedure goes in the other direction. Specifically, we can start with a nonnegative function $h \in L^1(\mathbf{R})$ such that $\int_{-\infty}^{\infty} h \, d\lambda = 1$. We use h to define a probability measure on $(\mathbf{R}, \mathcal{B})$ and then consider the identity random variable X on $\mathbf{R}$. The function h that we started with is then the density function of X. The following result formalizes this procedure and gives formulas for the mean and standard deviation in terms of the density function h.

12.33 ***mean and variance of random variable generated by density function***

Suppose $h \in L^1(\mathbf{R})$ is such that $\int_{-\infty}^{\infty} h \, d\lambda = 1$ and $h(x) \geq 0$ for almost every $x \in \mathbf{R}$. Let P be the probability measure on $(\mathbf{R}, \mathcal{B})$ defined by

$$P(B) = \int_B h \, d\lambda.$$

Let X be the random variable on $(\mathbf{R}, \mathcal{B})$ defined by $X(x) = x$ for each $x \in \mathbf{R}$. Then h is the density function of X. Furthermore, if $X \in \mathcal{L}^1(P)$ then

$$EX = \int_{-\infty}^{\infty} x h(x) \, d\lambda(x),$$

and if $X \in \mathcal{L}^2(P)$ then

$$\sigma^2(X) = \int_{-\infty}^{\infty} x^2 h(x) \, d\lambda(x) - \Bigl(\int_{-\infty}^{\infty} x h(x) \, d\lambda(x)\Bigr)^2.$$

Proof The equation $\tilde{X}(s) = \int_{-\infty}^{s} h\,d\lambda$ holds by the definitions of $\tilde{X}$ and P. Thus h is the density function of X.

Our definition of P to equal $h\,d\lambda$ implies that $\int_{-\infty}^{\infty} f\,dP = \int_{-\infty}^{\infty} fh\,d\lambda$ for all $f \in \mathcal{L}^1(P)$ [see Exercise 5 in Section 9A]. Thus the formula for the mean EX follows immediately from the definition of EX, and the formula for the variance $\sigma^2(X)$ follows from 12.20. ■

The following example illustrates the result above with a few especially useful choices of the density function h.

12.34 Example *density functions*

- Suppose $h = 1_{[0,1]}$. This density function h is called the *uniform density* on $[0,1]$. In this case, $P(B) = \lambda(B \cap [0,1])$ for each Borel set $B \subset \mathbf{R}$. For the corresponding random variable $X(x) = x$ for $x \in \mathbf{R}$, the distribution function $\tilde{X}$ is given by the formula
$$\tilde{X}(s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } 0 < s < 1, \\ 1 & \text{if } s \ge 1. \end{cases}$$
The formulas in 12.33 show that $EX = \frac{1}{2}$ and $\sigma(X) = \frac{1}{2\sqrt{3}}$.

- Suppose $\alpha > 0$ and
$$h(x) = \begin{cases} 0 & \text{if } x < 0, \\ \alpha e^{-\alpha x} & \text{if } x \ge 0. \end{cases}$$
This density function h is called the *exponential density* on $[0,\infty)$. For the corresponding random variable $X(x) = x$ for $x \in \mathbf{R}$, the distribution function $\tilde{X}$ is given by the formula
$$\tilde{X}(s) = \begin{cases} 0 & \text{if } s < 0, \\ 1 - e^{-\alpha s} & \text{if } s \ge 0. \end{cases}$$
The formulas in 12.33 show that $EX = \frac{1}{\alpha}$ and $\sigma(X) = \frac{1}{\alpha}$.

- Suppose
$$h(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
for $x \in \mathbf{R}$. This density function is called the *standard normal density*. For the corresponding random variable $X(x) = x$ for $x \in \mathbf{R}$, we have $\tilde{X}(0) = \frac{1}{2}$. For general $s \in \mathbf{R}$, no formula exists for $\tilde{X}(s)$ in terms of elementary functions. However, the formulas in 12.33 show that $EX = 0$ and (with the help of some calculus) $\sigma(X) = 1$.

Weak Law of Large Numbers

Families of random variables all of which look the same in terms of their distribution functions get a special name, as we see in the next definition.

12.35 Definition ***identically distributed; i.i.d.***

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

- A family of random variables on $(\Omega, \mathcal{F})$ is called *identically distributed* if all the random variables in the family have the same distribution function.
- More specifically, a family $\{X_k\}_{k \in \Gamma}$ of random variables on $(\Omega, \mathcal{F})$ is called *identically distributed* if

$$P(X_j \leq s) = P(X_k \leq s)$$

  for all $j, k \in \Gamma$.
- A family of random variables that is independent and identically distributed is said to be *independent identically distributed*, often abbreviated as *i.i.d.*

12.36 Example *family of random variables for decimal digits is i.i.d.*

Consider the probability space $([0,1], \mathcal{B}, P)$, where $\mathcal{B}$ is the collection of Borel subsets of the interval $[0,1]$ and P is Lebesgue measure on $([0,1], \mathcal{B})$. For $k \in \mathbf{Z}^+$, define a random variable $X_k \colon [0,1] \to \mathbf{R}$ by

$$X_k(\omega) = k^{\text{th}}\text{-digit in decimal expansion of } \omega,$$

where for those numbers ω that have two different decimal expansions we use the one that does not end in an infinite string of 9s.

Notice that $P(X_k \leq \pi) = 0.4$ for every $k \in \mathbf{Z}^+$. More generally, the family $\{X_k\}_{k \in \mathbf{Z}^+}$ is identically distributed, as you should verify.

The family $\{X_k\}_{k \in \mathbf{Z}^+}$ is also independent, as you should verify. Thus $\{X_k\}_{k \in \mathbf{Z}^+}$ is an i.i.d. family of random variables.

Identically distributed random variables have the same expectation and the same standard deviation, as the next result shows.

12.37 ***identically distributed random variables have same mean and variance***

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\{X_k\}_{k \in \Gamma}$ is an identically distributed family of random variables in $\mathcal{L}^2(P)$. Then

$$EX_j = EX_k \quad \text{and} \quad \sigma(X_j) = \sigma(X_k)$$

for all $j, k \in \Gamma$.

Proof Suppose $j \in \mathbf{Z}^+$. Let $f_1, f_2, \ldots$ be the sequence of simple functions converging pointwise to X_j as constructed in the proof of 2.89. The Dominated Convergence Theorem (3.31) implies that $EX_j = \lim_{n\to\infty} Ef_n$. Because of how each f_n is constructed, each Ef_n depends only on n and the numbers $P(c \leq X_j < d)$ for $c < d$. However,

$$P(c \leq X_j < d) = \lim_{m\to\infty}\Big(P\big(X_j \leq d - \tfrac{1}{m}\big) - P\big(X_j \leq c - \tfrac{1}{m}\big)\Big)$$

for $c < d$. Because $\{X_k\}_{k\in\Gamma}$ is an identically distributed family, the numbers above on the right are independent of j. Thus $EX_j = EX_k$ for all $j, k \in \mathbf{Z}^+$.

Apply the result from the paragraph above to the identically distributed family $\{X_k{}^2\}_{k\in\Gamma}$ and use 12.20 to conclude that $\sigma(X_j) = \sigma(X_k)$ for all $j, k \in \Gamma$. ■

The next result has the nicely intuitive interpretation that if we repeat a random process many times, then the probability that the average of our results differs from our expected average by more than any fixed positive number ε has limit 0 as we increase the number of repetitions of the process.

12.38 *Weak Law of Large Numbers*

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\{X_k\}_{k\in\mathbf{Z}^+}$ is an i.i.d. family of random variables in $\mathcal{L}^2(P)$, each with expectation μ. Then

$$\lim_{n\to\infty} P\Big(\Big|\big(\tfrac{1}{n}\sum_{k=1}^{n} X_k\big) - \mu\Big| \geq \varepsilon\Big) = 0$$

for all $\varepsilon > 0$.

Proof Because the random variables $\{X_k\}_{k\in\mathbf{Z}^+}$ all have the same expectation and same standard deviation, by 12.37 there exist $\mu \in \mathbf{R}$ and $s \in [0, \infty)$ such that

$$EX_k = \mu \quad \text{and} \quad \sigma(X_k) = s$$

for all $k \in \mathbf{Z}^+$. Thus

12.39 $$E\Big(\frac{1}{n}\sum_{k=1}^{n} X_k\Big) = \mu \quad \text{and} \quad \sigma^2\Big(\frac{1}{n}\sum_{k=1}^{n} X_k\Big) = \frac{1}{n^2}\sigma^2\Big(\sum_{k=1}^{n} X_k\Big) = \frac{s^2}{n},$$

where the last equality follows from 12.22 (this is where we use the independent part of the hypothesis).

Now suppose $\varepsilon > 0$. In the special case where $s = 0$, all the X_k are almost surely equal to the same constant function and the desired result clearly holds. Thus we assume $s > 0$. Let $t = \sqrt{n}\varepsilon/s$ and apply Chebyshev's inequality (12.21) with this value of t to the random variable $\frac{1}{n}\sum_{k=1}^{n} X_k$, using 12.39 to get

$$P\Big(\Big|\big(\frac{1}{n}\sum_{k=1}^{n} X_k\big) - \mu\Big| \geq \varepsilon\Big) \leq \frac{s^2}{n\varepsilon^2}.$$

Taking the limit as $n \to \infty$ of both sides of the inequality above gives the desired result. ■

EXERCISES 12

1 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A \in \mathcal{F}$. Prove that A and $\Omega \setminus A$ are independent if and only if $P(A) = 0$ or $P(A) = 1$.

2 Suppose P is Lebesgue measure on $[0, 1]$. Give an example of two disjoint Borel subsets sets A and B of $[0, 1]$ such that $P(A) = P(B) = \frac{1}{2}$, $[0, \frac{1}{2}]$ and A are independent, and $[0, \frac{1}{2}]$ and B are independent.

3 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$. Prove that the following are equivalent:

- A and B are independent events.
- A and $\Omega \setminus B$ are independent events.
- $\Omega \setminus A$ and B are independent events.
- $\Omega \setminus A$ and $\Omega \setminus B$ are independent events.

4 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\{A_k\}_{k \in \Gamma}$ is a family of events. Prove the family $\{A_k\}_{k \in \Gamma}$ is independent if and only if the family $\{\Omega \setminus A_k\}_{k \in \Gamma}$ is independent.

5 Give an example of a probability space $(\Omega, \mathcal{F}, P)$ and events A, B_1, B_2 such that A and B_1 are independent, A and B_2 are independent, but A and $B_1 \cup B_2$ are not independent.

6 Give an example of a probability space $(\Omega, \mathcal{F}, P)$ and events A_1, A_2, A_3 such that A_1 and A_2 are independent, A_1 and A_3 are independent, and A_2 and A_3 are independent, but the family A_1, A_2, A_3 is not independent.

7 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, $A \in \mathcal{F}$, and $B_1 \subset B_2 \subset \cdots$ is an increasing sequence of events such that A and B_n are independent events for each $n \in \mathbf{Z}^+$. Show that A and $\bigcup_{n=1}^{\infty} B_n$ are independent.

8 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\{A_t\}_{t \in \mathbf{R}}$ is an independent family of events such that $P(A_t) < 1$ for each $t \in \mathbf{R}$. Prove that there exists a sequence $t_1, t_2, \ldots$ in $\mathbf{R}$ such that $P\bigl(\bigcap_{n=1}^{\infty} A_{t_n}\bigr) = 0$.

9 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $B_1, \ldots, B_n \in \mathcal{F}$ are such that $P(B_1 \cap \cdots \cap B_n) > 0$. Prove that

$$P(A \cap B_1 \cap \cdots \cap B_n) = P(B_1) \cdot P_{B_1}(B_2) \cdots P_{B_1 \cap \cdots \cap B_{n-1}}(B_n) \cdot P_{B_1 \cap \cdots \cap B_n}(A)$$

for every event $A \in \mathcal{F}$.

10 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A \in \mathcal{F}$ is an event such that $0 < P(A) < 1$. Prove that

$$P(B) = P_A(B) \cdot P(A) + P_{\Omega \setminus A}(B) \cdot P(\Omega \setminus A)$$

for every event $B \in \mathcal{F}$.

11 Give an example of a probability space $(\Omega, \mathcal{F}, P)$ and $X, Y \in \mathcal{L}^2(P)$ such that $\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y)$ but X and Y are not independent random variables.

12 Prove that if X and Y are random variables (possibly on two different probability spaces) and $\tilde{X} = \tilde{Y}$, then $P_X = P_Y$.

13 Suppose $H\colon \mathbf{R} \to (0,1)$ is a continuous one-to-one function satisfying conditions (a) through (d) of 12.29. Show that the function $X\colon (0,1) \to \mathbf{R}$ produced in the proof of 12.29 is the inverse function of H.

14 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and X is a random variable. Prove that the following are equivalent:

- $\tilde{X}$ is a continuous function on $\mathbf{R}$.
- $\tilde{X}$ is a uniformly continuous function on $\mathbf{R}$.
- $P(X = t) = 0$ for every $t \in \mathbf{R}$.
- $(\tilde{X} \circ X)\tilde{}(s) = s$ for all $s \in \mathbf{R}$.

15 Suppose $\alpha > 0$ and $h(x) = \begin{cases} 0 & \text{if } x < 0, \\ \alpha^2 x e^{-\alpha x} & \text{if } x \geq 0. \end{cases}$

Let $P = h\,d\lambda$ and let X be the random variable defined by $X(x) = x$ for $x \in \mathbf{R}$.

(a) Verify that $\int_{-\infty}^{\infty} h\,d\lambda = 1$.

(b) Find a formula for the distribution function $\tilde{X}$.

(c) Find a formula (in terms of α) for EX.

(d) Find a formula (in terms of α) for $\sigma(X)$.

16 Suppose $\mathcal{B}$ is the σ-algebra of Borel subsets of $[0,1)$ and P is Lebesgue measure on $([0,1], \mathcal{B})$. Let $\{e_k\}_{k \in \mathbf{Z}^+}$ be the family of functions defined by the fourth bullet point of Example 8.51 (notice that $k = 0$ is excluded). Show that the family $\{e_k\}_{k \in \mathbf{Z}^+}$ is an i.i.d.

17 Suppose $\mathcal{B}$ is the σ-algebra of Borel subsets of $(-\pi, \pi]$ and P is Lebesgue measure on $((-\pi, \pi], \mathcal{B})$ divided by 2π. Let $\{e_k\}_{k \in \mathbf{Z} \setminus \{0\}}$ be the family of trigonometric functions defined by the third bullet point of Example 8.51 (notice that $k = 0$ is excluded).

(a) Show that $\{e_k\}_{k \in \mathbf{Z} \setminus \{0\}}$ is not an independent family of random variables.

(b) Show that $\{e_k\}_{k \in \mathbf{Z} \setminus \{0\}}$ is an identically distributed family.

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The chapters of this book on linear operators on Hilbert spaces, Fourier analysis, and probability only introduce these huge subjects, all of which have a vast literature. This bibliography gives book recommendations for readers who want to go into these topics in more depth.

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Notation Index

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- This book was typeset in pdfLaTeX by the author, who wrote the LaTeX code to implement the book's design. The pdfLaTeX software was developed by Hàn Thế Thành.
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- The main text font in this book is Nimbus Roman No. 9 L, created by URW as a legal clone of Times, which was designed for the British newspaper *The Times* in 1931.
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- The san serif font used for chapter titles, section titles, and subsection titles is NimbusSanL.
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